THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2009
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: TWO hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (17 marks)

(a) Find a solution to the initial value problem

$$\frac{dy}{dt} = \frac{y}{t} - 1, \quad y(1) = 2.$$

You may assume that t is positive for this part of the question.

(b) Find a one-parameter family of solutions to the differential equation

$$\frac{dy}{dt} = \frac{te^t}{y}.$$

(c) Find the general solution to the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{dy}{dt} = 2t.$$

2. (17 marks) This question is about the one-parameter family of differential equations

$$\frac{dy}{dt} = y^2 + 2y - a.$$

(a) For the case $a = 0$, find all equilibrium solutions and determine their type (e.g., sink, source). Sketch the phase line.

(b) Repeat (a) for the case $a = -4$.

(c) Now let a vary.

(i) Locate the equilibrium solutions and determine their type for all values of a including any bifurcation values.

(ii) Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram.

3. (15 marks) Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & a \\ -2 & 0 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) For the choice $a = 1$

(i) Find the general solution to the differential equation. Express your answer in terms of real-valued functions.

(ii) Sketch the corresponding phase portrait.

(b) For the choice $a = -2$

(i) Find the general solution to the differential equation. Express your answer in terms of real-valued functions.

(ii) Sketch the corresponding phase portrait.

(c) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a saddle.

4. (16 marks)

(a) Consider the differential equation

$$\frac{dx}{dt} = -x^2 + t.$$

There are two copies of this direction field for this differential equation on the yellow answer sheet attached to the back of this question paper.

(i) Use one copy of the direction field on the yellow answer sheet to sketch at least four representative solutions to the differential equation, including the solution that satisfies the initial condition $x(1) = -1$.

(ii) Use the other copy of the direction field on the yellow answer sheet to show what would be obtained if Euler's method with stepsize $h = 1$ was used to compute an approximation (at final time $t = 3$) to the solution that satisfies the initial condition $x(1) = -1$. You do not need to do any calculations to do this part of the question; just use the information on the direction field.

(b) Use Improved Euler's method with stepsize $h = 1$ to compute an approximate value of the solution to the initial value problem

$$\frac{dx}{dt} = -x^2 + t,\ x(1) = -1$$

at final time $t = 3$. Show all your working.

(c) A different numerical method is used to estimate $x(2)$ for various choices of stepsize. The following results are obtained.

Number of steps	approximate x(2)
1	-0.1881510417
2	-0.1131418135
4	-0.1037212734
8	-0.1029488697
16	-0.1028930652
32	-0.1028893052
64	-0.1028890611

(i) Use these results to estimate $x(2)$ accurate to 5 decimal places.

(ii) Estimate the errors in the approximation obtained using 4 steps and 8 steps.

(iii) Calculate the effective order of the method using the error estimates in (ii).

(iv) Which numerical method do you think might have been used to get these results? Give a reason for your answer.

5. (25 marks) Consider the following system of equations:

$$\frac{dx}{dt} = x(x+y),$$
$$\frac{dy}{dt} = y - x^2 + 2, \quad x \geq 0.$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find all equilibrium solutions (with $x \geq 0$) and determine their type (e.g., spiral source, saddle). For each equilibrium you find, draw a phase portrait showing the behaviour of solutions near that equilibrium.

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system. Your phase portrait should show the behaviour of solutions near the equilibria, and should show various solution curves *including* those passing through the following initial conditions:

(i) $(x(0), y(0)) = (2, 0)$;

(ii) $(x(0), y(0)) = (0, -1)$.

Make sure you show clearly where solution curves go as t increases and decreases.

6. (10 marks) The outbreak of a disease such as influenza within a population can be modeled with the system of equations:

$$\frac{dS}{dt} = -aSI,$$
$$\frac{dI}{dt} = aSI - bI,$$
$$\frac{dR}{dt} = bI,$$

where a and b are positive constants. In this model $S(t)$ is the number of susceptible people in the population (these are people who have not had influenza and have no immunity to it). $I(t)$ is the number people in the population who are infected with influenza. $R(t)$ is the number of recovered people in the population (these are people who no longer have influenza and are now immune to it). In this model, nobody dies from the disease. The variable t is time.

(a) Briefly describe the physical significance of each term in the model.

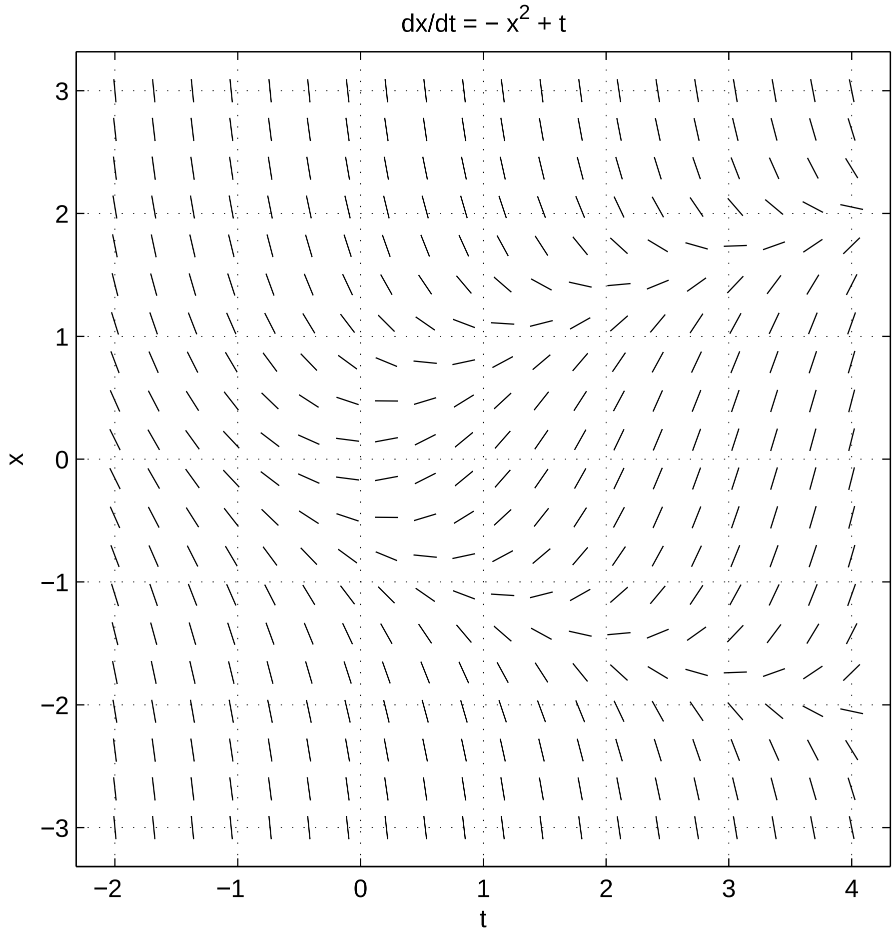
(b) One way used to prevent the spread of influenza is by vaccinating susceptible people in the population. This works because people who get vaccinated become immune to the disease. Suggest how you could modify the model to include this effect.

(c) Briefly describe (i.e., in one or two paragraphs) some methods you could use to analyse these equations to get information about solutions to the model.

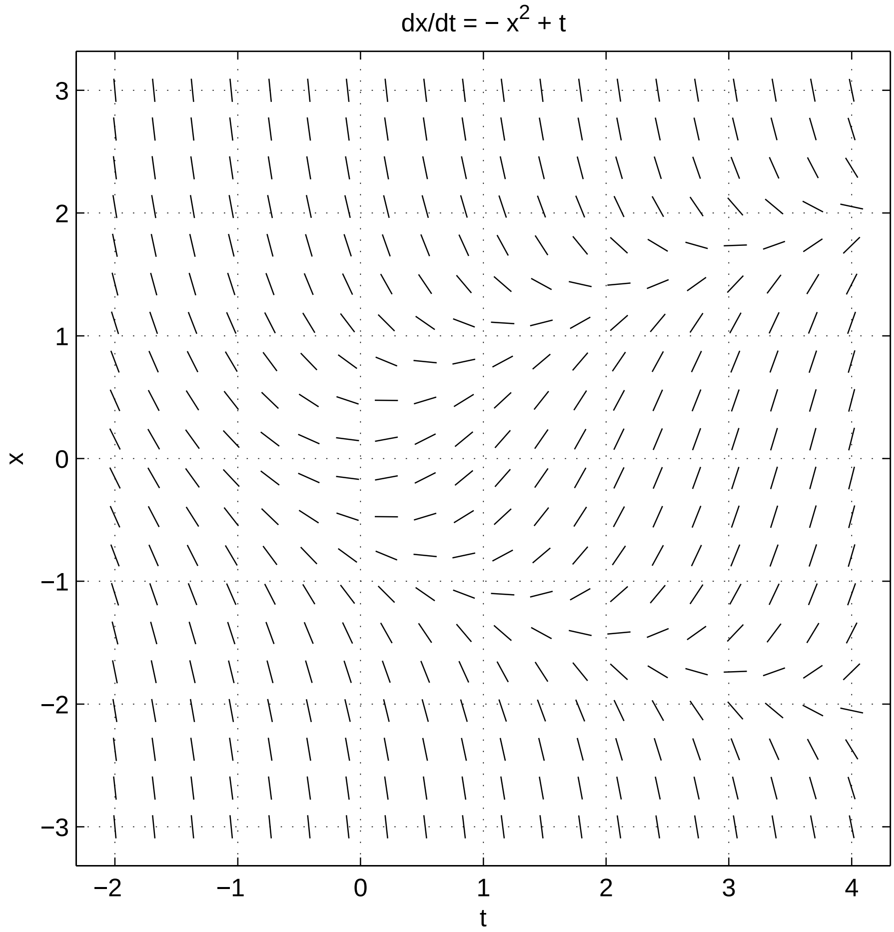
Candidate's Name: ____________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 4(a)(i)



Answer sheet for Question 4(a)(ii)



Answer sheet for Question 5