Maths 260 2009S Mid-semester Test

Solutions.

1. (6 marks) Find a one-parameter family of solutions to the differential equation

$$\frac{dy}{dt} = 2ty + 3e^{t^2}.$$

This is a linear eqn. Write in standard form to calculate integrating factor

$$\frac{dy}{dt} - 2ty = 3e^{t^2}$$

Then $\mu = \exp\left(\int -2t\,dt\right) = e^{-t^2}$

so $e^{-t^2}\dfrac{dy}{dt} - 2te^{-t^2}y = 3$

(multiplying through by $\mu$)

$$\Rightarrow \quad \frac{d}{dt}\left(ye^{-t^2}\right) = 3$$

$$\Rightarrow \quad ye^{-t^2} = \int 3\,dt$$

$$= 3t + c$$

$$\Rightarrow \quad y = 3te^{t^2} + ce^{t^2}$$

2. (7 marks)

(a) Use Euler's method with stepsize $h = 1$ to compute an approximate value of the solution to the initial value problem

$$\frac{dy}{dt} = yt - y^2, \quad y(0) = -1$$

at final time $t = 2$. Show all your working.

First step $y_0 = -1$ $t_0 = 0$ $h = 1$

$f(t_0, y_0) = -1 \times 0 - (-1)^2 = -1$

$y_1 = y_0 + h f(t_0, y_0)$

$= -1 + 1 \times -1 = -2$

Second step: $y_1 = -2$ $t_1 = 1$ $h = 1$

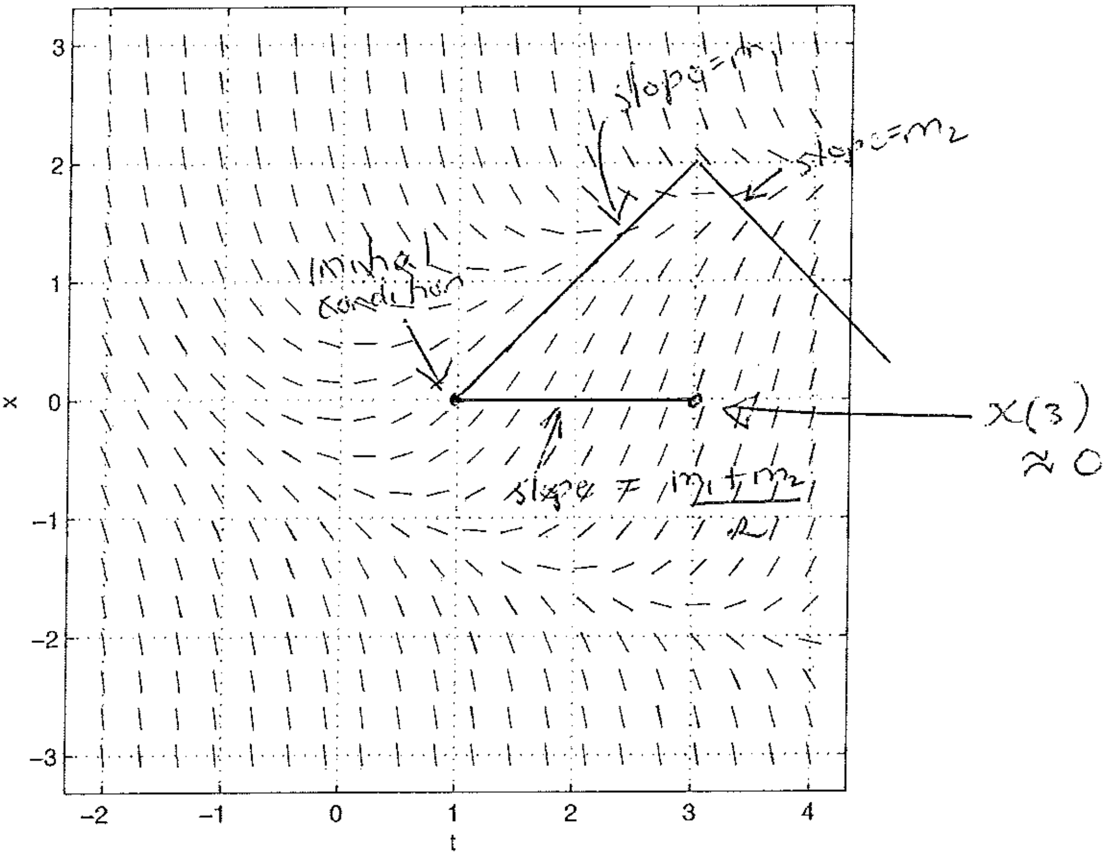
$f(t_1, y_1) = -2 \times 1 - (-2)^2 = -6$

$y_2 = y_1 + h f(t_1, y_1)$

$= -2 + 1 \times -6 = -8$

so $y(2) \approx -8$ according to this method

(b) The direction field for a differential equation is shown below.




On the direction field, sketch the numerical solution that would be obtained if one step of Improved Euler's method with stepsize $h = 2$ was used to solve the differential equation with initial condition $x(1) = 0$.

You do not need to do any calculations to answer this part of the question; just use the information on the direction field.

3. (15 marks) This question is about the one-parameter family of differential equations

$$\frac{dy}{dt} = (y-1)(y-k)$$

where $k$ is the parameter.

(a) Set $k = 0$.

  i. Find all equilibrium solutions and determine their type (e.g., sink, source).

  ii. Sketch the phase line.

(b) Repeat (a) for the case $k = 1$.

(c) Repeat (a) for the case $k = 2$.

(d) Now let $k$ vary.

  i. Locate the equilibrium solutions and determine their type for all values of $k$, including any bifurcation values.

  ii. Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram.

(a) $$\frac{dy}{dt} = y(y-1)$$

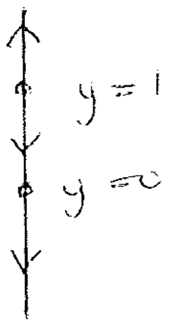
Equilibria are $y=0$ and $y=1$

$$\frac{\partial f}{\partial y} = 2y-1$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = -1 \implies y=0 \text{ is a sink}$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = 1 \implies y=1 \text{ is a source}$$

Phase line:



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(b) $\dfrac{dy}{dt} = (y-1)^2$

Only one equilibrium, $y = 1$

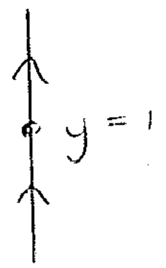
$\dfrac{\partial f}{\partial y} = 2(y-1)$

$\left.\dfrac{\partial f}{\partial y}\right|_{y=1} = 0 \Rightarrow$ linearisation tells us nothing about type of equilibrium.

However, note that $\dfrac{dy}{dt} = (y-1)^2 \geq 0$

with $\dfrac{dy}{dt} > 0$ if $y \neq 1$.

Thus the phase line is




from which it is seen that $y=1$ is a node.

(c) $\dfrac{dy}{dt} = (y-1)(y-2) = y^2 - 3y + 2$

Equilibria are $y=1$ and $y=2$

$\dfrac{\partial f}{\partial y} = 2y - 3$

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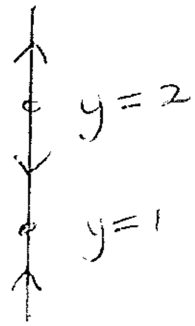
$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = -1 \quad \Rightarrow \quad y=1 \text{ is a sink}$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=2} = 1 \quad \Rightarrow \quad y=2 \text{ is a source.}$$

Phase line:




(d) $\dfrac{dy}{dt} = (y-1)(y-k) = y^2-(k+1)y+k$

Equilibria are $y=1$ (for all $k$) and $y=k$.

$$\frac{\partial f}{\partial y} = 2y-(k+1)$$

$$\left.\frac{\partial f}{\partial y}\right|_{y=1} = 2-k-1 = 1-k$$

so $y=1$ is a sink if $k>1$
$y=1$ is a source if $k<1$
(at $k=1$, $y=1$ is a node, by (b))

$$\left.\frac{\partial f}{\partial y}\right|_{y=k} = 2k-k-1 = k-1$$

$\Rightarrow$ $y=k$ is a source if $k>1$
$y=k$ is a sink if $k<1$

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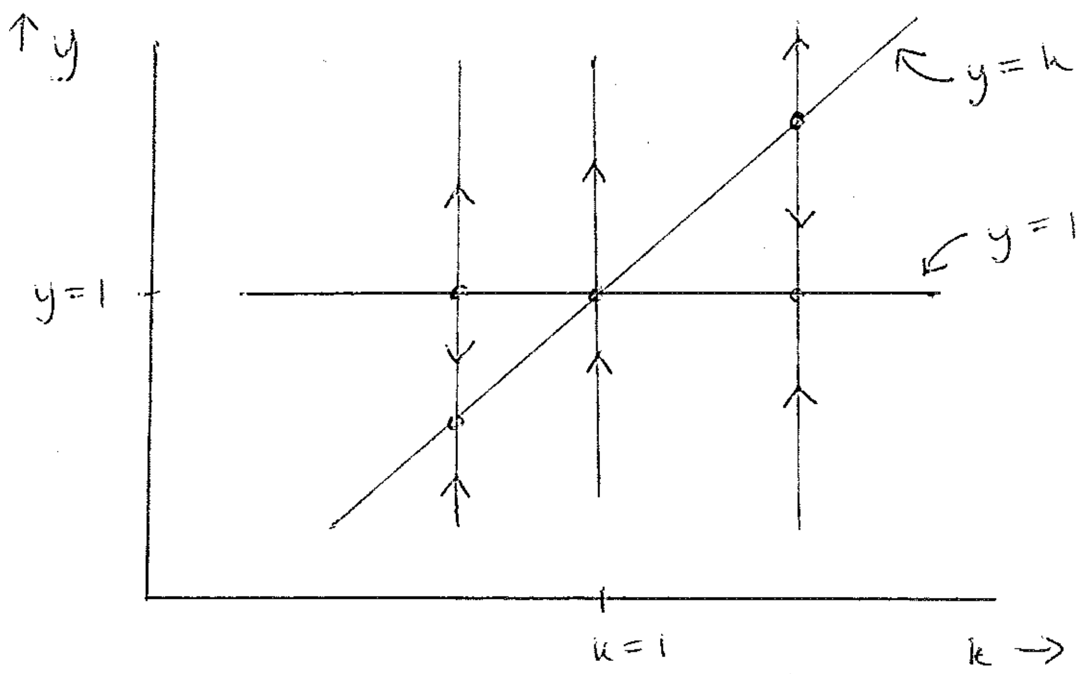
( $y=k$ is a node if $k=1$, from (b) above)

<u>Bifurcation diagram.</u>




Bifurcation at $k=1$ (transcritical bifurcation)

4. (7 marks) Consider the following system of differential equations

$$\frac{dY}{dt} = \begin{pmatrix} 1 & 0 \\ 2 & -2 \end{pmatrix} Y,$$

where

$$Y = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find all straight line solutions to this system of equations.

(b) Find the general solution to this system of equations. Your answer should contain two arbitrary constants.

(c) Find the solution that passes through $(x, y) = (1, 0)$ when $t = 0$.

(d) Sketch the phase portrait showing:

- all equilibrium solutions.
- all straight line solutions.
- the solution curve you found in part (c) above, for $t < 0$ as well as $t > 0$. Indicate on your sketch where $t = 0$.
- at least three other representative solution curves.

(a) Eigenvalues are 1 and $-2$ (since triangular matrix).

evecs

for $\lambda = 1$

$$\begin{pmatrix} 0 & 0 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2x = 3y$

Choose $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$\lambda = -2$

$$\begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x = 0$

Choose $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Straight line sol$^{n}$s are

$Y_1 = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t}$ $\qquad$ $Y_2 = c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$

(blank page for your working)

b) Gen. sol$^n$ is

$$\underline{Y} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

(c) $\underline{Y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

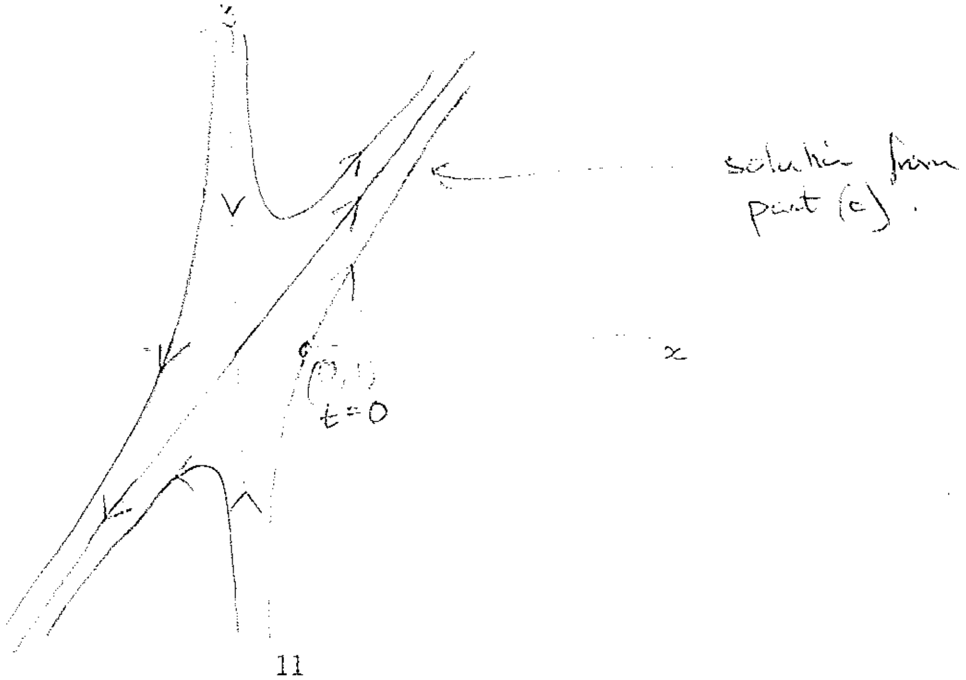
$$\Rightarrow \quad c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$3c_1 = 1 \quad \Rightarrow \quad c_1 = 1/3$$

$$2c_1 + c_2 = 0 \qquad c_2 = -2c_1 = -2/3$$

sol$^n$ is $\underline{Y} = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ -2/3 \end{pmatrix} e^{-2t}$

d)



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5. (5 marks) This question is about a mathematical model for repayment of a bank loan.

Sandra has a loan from a bank. The bank charges her a fixed interest rate for the loan and she pays the bank a fixed amount of money towards her loan each week.

Sandra uses the following differential equation to model the growth of her loan:

$$\frac{dL}{dt} = 0.055L - 260, \quad L(0) = 10.$$

The variable $L$ gives the amount of her loan in thousands of dollars (so that $L = 1$ means that her loan is \$1000) and $t$ measures time in years. Use this information to answer the following questions.

(a) How much is her loan to start with?

(b) What is the interest rate on the loan?

(c) How much does she pay to the bank each week?

(d) In one or two sentences, say how you could work out how long it will take Sandra to pay back her loan. You do not have to do any calculations to answer this part of the question.

(a) \$10,000

(b) 5.5 %

(c) \$260 000 per year

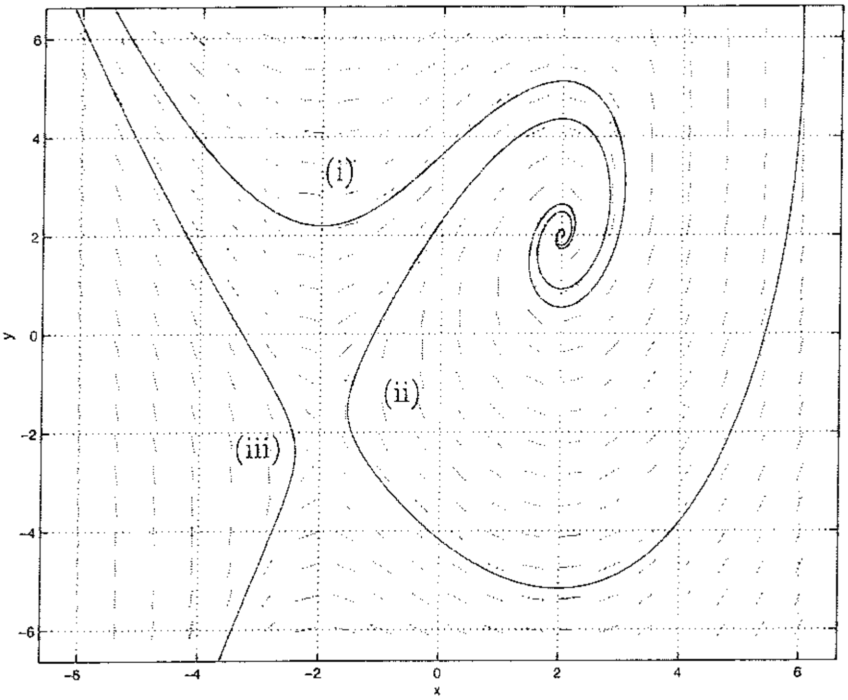
or $\approx \frac{260000}{52} = \$5000$ per week

(this is a lot! probably a bad model)

(d) Either solve the IVP explicitly (this can be done since the model is separable) to get $L(t)$, then find $t_0$ such that $L(t_0) = 0$

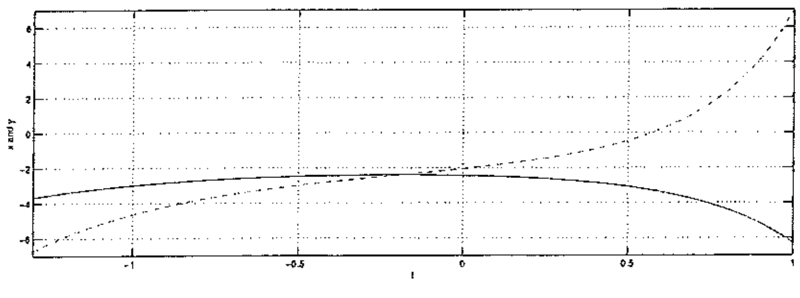
or use matlab (dfield) to solve the IVP numerically and find the time at which $L(t) = 0$.

6. (5 marks) The following figure shows the direction field and some solutions for a two-dimensional system of autonomous ODEs.

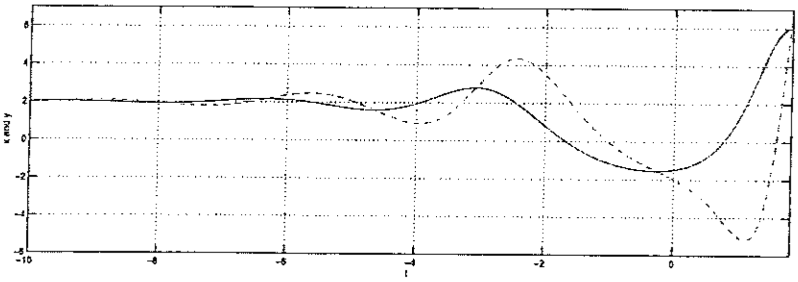



The following three figures (a), (b) and (c) show plots of $x(t)$ and $y(t)$ against $t$ for the three solutions (i), (ii) and (iii) shown above, in some order. Match up each of the plots (a), (b) and (c) with the correct solutions (i), (ii) or (iii) shown in the phase portrait.

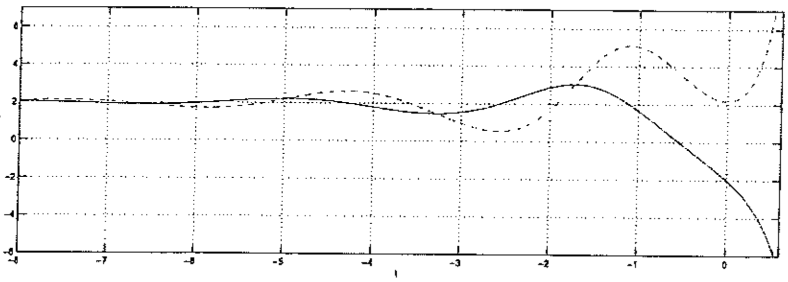
(a)



(b)




(c)




(a) is (iii) [it has no "wiggles", one coordinate ($x$) is always $< 0$, the other increases ].

(b) is (ii) [wiggles, so must be a sol^n in the spiral, both co-ords go both above and below zero ]

(c) is (i) [wiggles, and one co-ord stays $> 0$ ]