Maths 260 Lecture 32

- **Topic for today:** Nonhomogeneous higher order DEs
- **Reading for this lecture:** BDH Section 4.1, 4.2
- **Suggested exercises:** BDH Section 4.1; 1, 3, 7, 11 and Section 4.2; 1, 3, 9, 13
- **Reading for next lecture:** BDH Section 4.3
- **Today's handouts:** None

Nonhomogeneous higher order linear DEs

An nth order linear DE of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

is called nonhomogeneous if $f(t)$ is nonzero.

The function $f(t)$ is called the forcing function or nonhomogeneous term.

Example 1:

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = \sin t$$

models the behaviour of a mass/spring system subject to periodic forcing.

To solve a nonhomogeneous DE, we first solve the corresponding homogeneous equation and then combine this solution with a particular solution to the nonhomogeneous equation.

We saw this idea earlier when looking for solutions to first order linear systems.

For the nonhomogeneous DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

the corresponding homogeneous equation is:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

The extended linearity principle

- If y_h is a solution to the homogeneous DE and y_p is a particular solution to the nonhomogeneous DE then

$$y = y_h + y_p$$

  is also a solution to the nonhomogeneous DE.

- If y_c is the *general* solution to the homogeneous DE and y_p is a particular solution to the nonhomogeneous equation then

$$y = y_c + y_p$$

  is the *general* solution to the nonhomogeneous DE.

Verification of the extended linearity principle for the DE $y'' + py' + qy = f(t)$:

Example 2:

Show that $y_p = -2\mathrm{e}^{-3t}$ is a particular solution to the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2\mathrm{e}^{-3t}$$

Hence find the general solution to this equation.

Summary of method of solution for nonhomogeneous equations:

1. Find the general solution y_c to the corresponding homogeneous equation.
2. Find one solution to the nonhomogeneous equation (a particular solution y_p).
3. Add answers to steps 1 and 2 to get the general solution to the nonhomogeneous equation.
4. If trying to solve an IVP, use the initial conditions to determine constants in the general solution.

Finding a particular solution

We saw that when the harmonic oscillator is forced, solutions frequently mimic the forcing, at least in the long term.

We use this observation as the basis of a method for finding particular solutions to linear, constant coefficient DEs.

Example 3: Find a particular solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \mathrm{e}^t$$

Example 4:

Find a particular solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$$

Example 5:

Find a particular solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2t^2$$

Example 6:

Find a particular solution to the DE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \mathrm{e}^{-t}$$

We can formalise the guessing method used in these examples:

Method of undetermined coefficients: To find a particular solution to the DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

where $f(t)$ is

- a constant, or
- t^n for n a positive integer, or
- $\mathrm{e}^{\lambda t}$ for real nonzero λ, or
- $\sin(bt)$ or $\cos(bt)$, for b constant, or
- a finite product of terms like these

take the following steps:

Step 1: Write down the UC set, which is the set made up of the function f and all linearly independent functions obtained by repeated differentiation of f.

Step 2: If any of the functions in the UC set is also a solution to the homogeneous DE, multiply all functions in the set by t^k, where k is the smallest integer so that the modified UC set does not contain any solutions to the homogeneous DE.

Step 3: Find a particular solution to the DE by taking a linear combination of all the functions in the (possibly modified) UC set. Determine the unknown constants by substituting this linear combination into the DE.

Example 7:

Find a solution to the DE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = t$$

Example 8:

Find a solution to the IVP

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = te^{3t}, \quad y(0) = 0,\ y'(0) = 1$$

Try $y_p = k_1 t^2 \mathrm{e}^{3t} + k_2 t \mathrm{e}^{3t} = \mathrm{e}^{3t}(k_1 t^2 + k_2 t)$. Then

$$\begin{aligned} y_p' &= e^{3t}(2k_1 t + k_2) + 3\mathrm{e}^{3t}(k_1 t^2 + k_2 t) \\ &= e^{3t}(3k_1 t^2 + (3k_2 + 2k_1)t + k_2) \\ y_p'' &= \mathrm{e}^{3t}(6k_1 t + (3k_2 + 2k_1)) + 3\mathrm{e}^{3t}(3k_1 t^2 + (3k_2 + 2k_1)t + k_2) \\ &= \mathrm{e}^{3t}(9k_1 t^2 + (9k_2 + 12k_1)t + (6k_2 + 2k_1)) \end{aligned}$$

So

$$\begin{aligned} y_p'' - 2y_p' - 3y_p = {} & t^2\mathrm{e}^{3t}(9k_1 - 2 \times 3k_1 + 3k_1) \\ & + t\mathrm{e}^{3t}(9k_2 + 12k_1 - 2(3k_2 + 2k_1) - 3k_2) \\ & + \mathrm{e}^{3t}(6k_2 + 2k_1 - 2k_2) \\ = {} & t\mathrm{e}^{3t}(8k_1) + \mathrm{e}^{3t}(4k_2 + 2k_1) \end{aligned}$$

So we must have:

and the general solution is

$$y(t) = \frac{1}{8}t^2\mathrm{e}^{3t} - \frac{1}{16}t\mathrm{e}^{3t} + c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{3t}$$

Initial conditions: $y(0) = 0,\ y'(0) = 1$:

Important ideas from today

To solve a nonhomogeneous DE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f(t)$$

we first solve the corresponding homogeneous equation (i.e., with $f(t) = 0$) and then combine this solution with a particular solution to the nonhomogeneous equation.

To find a particular solution, we guess a solution with a similar form to $f(t)$.

Specifically, we

- Form a UC set consisting of f and all linearly independent functions obtained by repeated differentiation of f.
- If any of the functions in the UC set is also a solution to the homogeneous DE, multiply all functions in the set by t^k, where k is the smallest integer so that the modified UC set does not contain any solutions to the homogeneous DE.
- Find a particular solution to the DE by taking a linear combination of all the functions in the (possibly modified) UC set. Determine the unknown constants by substituting this linear combination into the DE.