THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2007
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: TWO hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (15 marks)

(a) Solve the initial value problem

$$\frac{dy}{dt} = 3t^2 y, \quad y(0) = 2.$$

(b) Find the general solution of

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = t.$$

2. (8 marks) The direction field for the differential equation

$$\frac{dy}{dt} = 2t - \cos\left(\tfrac{y}{2}\right)$$

is given on the yellow answer sheet attached to the back of the question paper. Using the initial condition $y(0) = 1$:

(a) draw on the direction field as accurately as you can a solution satisfying the initial condition and use it to find $y(2)$;

(b) draw on the direction field the Euler method with $h = 0.5$ to estimate $y(2)$;

(c) from (a) and (b) write down the error in the Euler method's estimate.

3. (16 marks) Construct a bifurcation diagram for the differential equation

$$\frac{dy}{dt} = y^2 - \mu.$$

Identify any value of μ at which there is a bifurcation. Show all your working.

4. (24 marks) Consider the following two systems of differential equations:

(a)

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{Y}$$

(b)

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{Y}$$

For each system:

(i) Determine the general solution. Express your answer in terms of real-valued functions.

(ii) Carefully sketch the phase portrait.

(iii) Describe the long term behaviour of *all* the solutions.

5. (22 marks) Consider the following system of equations:

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = x - y^2$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find the two equilibrium solutions and determine their type.

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system.Your phase portrait should show various solution curves including those passing through the initial conditions

(i) $(x(0), y(0)) = (1.5, 0)$;

(ii) $(x(0), y(0)) = (1.5, 1)$;

(iii) $(x(0), y(0)) = (-0.5, -0.5)$.

Make sure you show clearly where these solution curves go as $t \to \infty$.

6. (15 marks)

(a) A mass attached to one end of a spring and oscillating on a horizontal table is modelled by the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

Explain each of the three terms in the differential equation. Make sure you say what the constants m, b and k represent, and whether there are any conditions on the signs or sizes of the constants.

(b) If the spring in (a) is now forced, it can be modelled by the differential equation

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \cos\omega t.$$

(i) Explain what the term $\cos\omega t$ models.

(ii) In the case when the model is

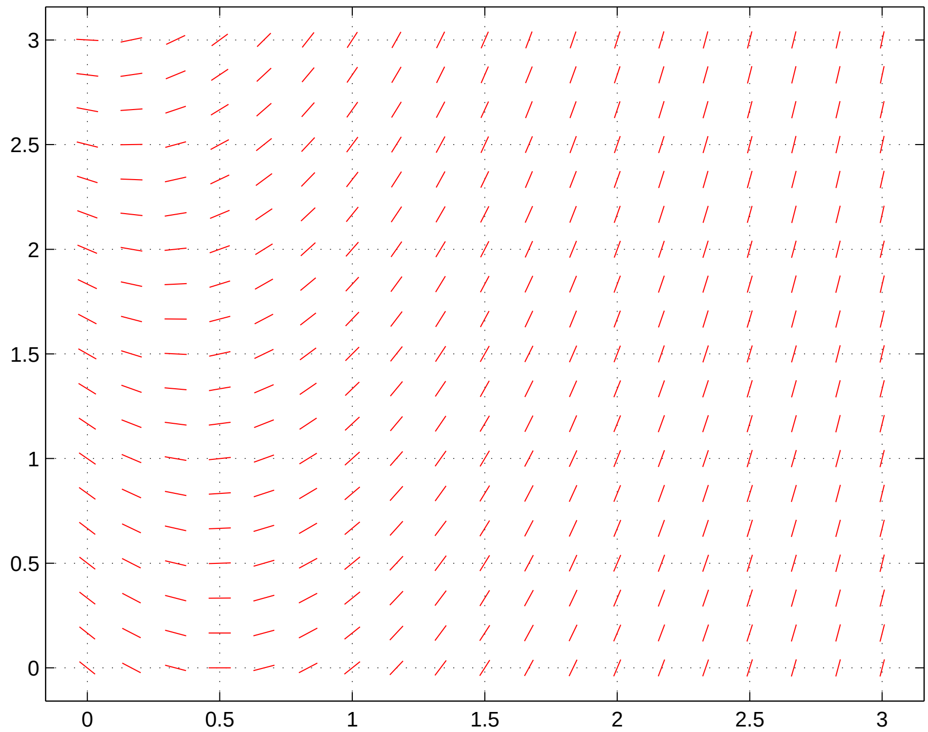
$$\frac{d^2x}{dt^2} + 4x = \cos 2t,$$

explain *why* the particular solution will contain terms in $t\cos 2t$ and $t\sin 2t$. You do **NOT** need to *find* the particular solution.

Candidate's Name: ______________________________ ID No: ______________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

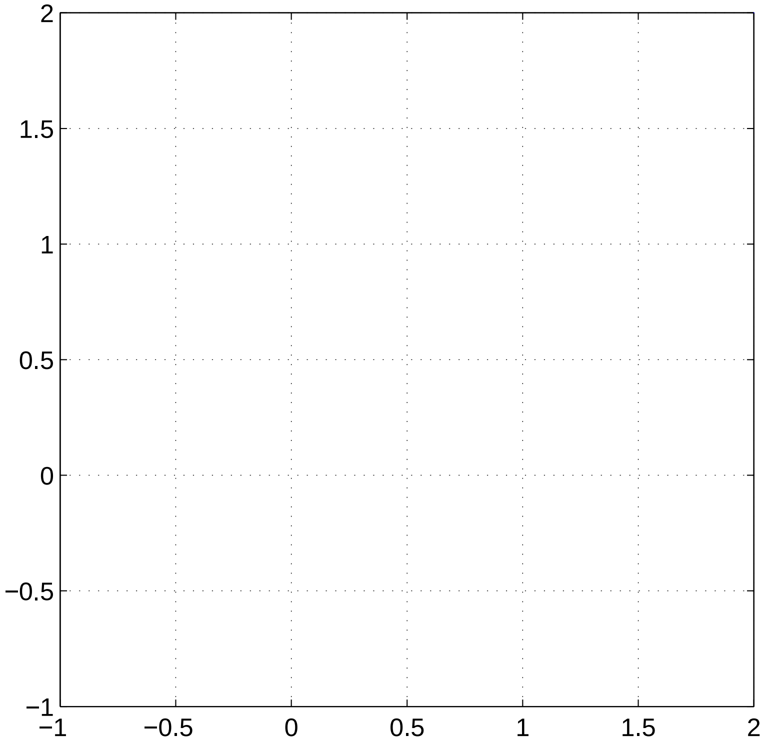
Answer sheet for Question 2



Candidate's Name: ________________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 6



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