Maths 260 Assignment 3

September 15, 2009 Due: 4pm, Tuesday September 29th, 2009

- Put your completed assignment in the appropriate box in the basement of the Maths/Physics building **before** 4pm on the date due.
- Late assignments or assignments placed in the wrong box will not be marked.
- Your assignment must be accompanied by a blue Mathematics Department coversheet. Copies of the coversheet are available from a box in the basement.

1. (22 marks) Consider the following linear system of differential equations

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY},$$

where $\mathbf{A}$ is given below. For each of the following matrices $\mathbf{A}$, (i) classify the equilibrium point at $\mathbf{Y} = \mathbf{0}$; and (ii) sketch the phase portrait near $\mathbf{Y} = \mathbf{0}$. Make sure to show a selection of representative solutions as well as the straight line solutions, and show the direction of the flow with arrows. *You may check your solutions using pplane, but you will not receive full marks unless you show all your workings to back up your sketches.*

(a) $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$

(d) $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix}$

2. (7 marks) Consider the IVP

$$\begin{aligned} \frac{dx}{dt} &= 2xy, \\ \frac{dy}{dt} &= x - t, \end{aligned}$$

with initial condition $x(0) = 1, y(0) = 0$.

(a) Use Euler's method with $h = 0.2$ to estimate the solution at final time $t = 0.6$.

(b) Plot (by hand) the graphs of $x(t)$ vs t and $y(t)$ vs t for your approximate solution.

(c) Describe (in one or two sentences) how you would check if your solution was accurate. You do not need to do any computation for this part of the question.

3. (6 marks) Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find the general solution to the differential equation.

(b) Sketch the phase portrait, showing various solutions including the straight line solutions, and the solution satisfying $(x(0), y(0)) = (0, 1)$.

4. (9 marks) Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -2y + z, \\ \dot{z} &= -y - 2z. \end{aligned}$$

(a) Write the equations in matrix form.

(b) Using the matrix form you found in (a), find the general solution in terms of real-valued functions.

(c) What type of equilibrium point is the origin?

(d) What is the long term behaviour of solutions?

5. (7 marks)

(a) By direct computation of the product $\mathbf{A} \cdot \mathbf{v}$, show that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

is an eigenvector of the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix},$$

and find the corresponding eigenvalue.

(b) Hence find a complex solution to the equation

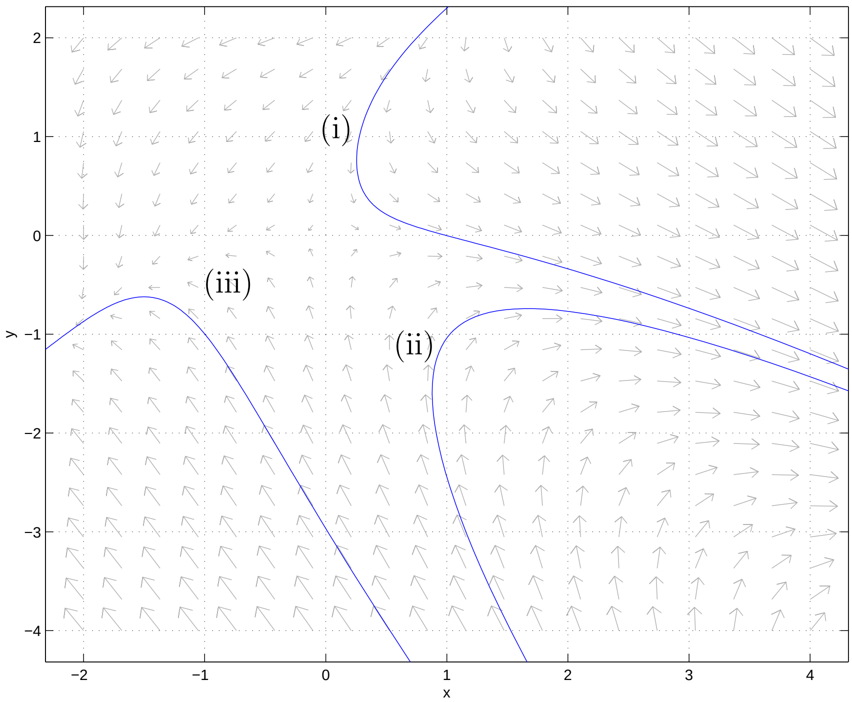
$$\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y}, \tag{1}$$

where $\mathbf{A}$ is given above, and $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$.

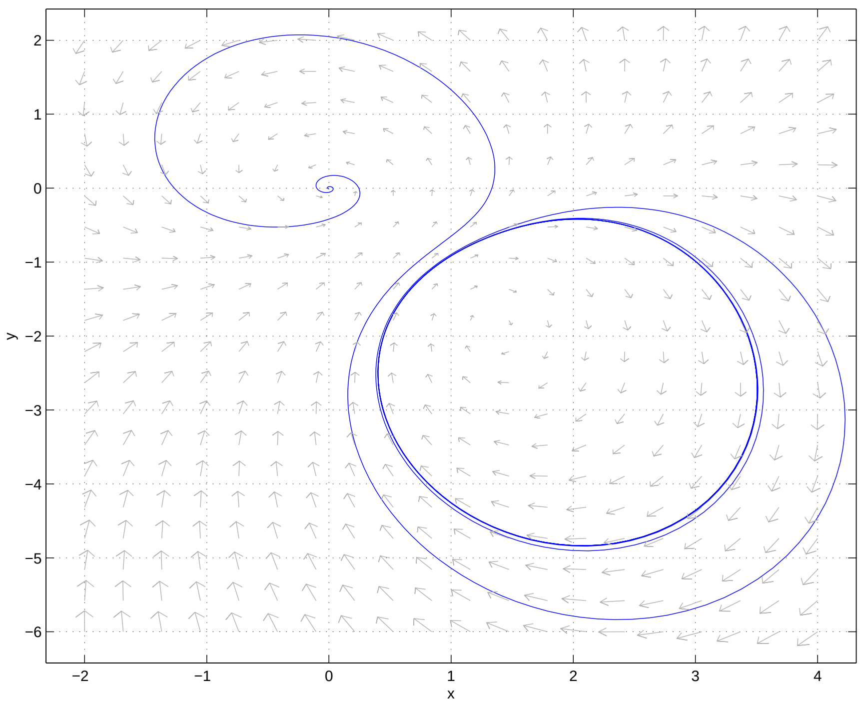
(c) By writing your complex solution in real and imaginary parts, find the general solution to the differential equation (1) in terms of real-valued functions.

6. (9 marks) The following pictures show vector fields for two-dimensional autonomous systems of ODEs, and some solutions:

(a)



(b)



For each solution shown, that is, for (a) (i), (ii) and (iii) and for (b), sketch a graph of $x(t)$ vs t and $y(t)$ vs t (for each solution, show both $x(t)$ and $y(t)$ on the same axes).