Maths 260 Lecture 8

- **Topics for today:**
  The phase line
- **Reading for this lecture:** BDH Section 1.6, pp 76-85
- **Suggested exercises:** BDH Section 1.6, #23,25,27,29.
- **Reading for next lecture:** BDH Section 1.6, pp 86-91
- **Today's handouts:** None

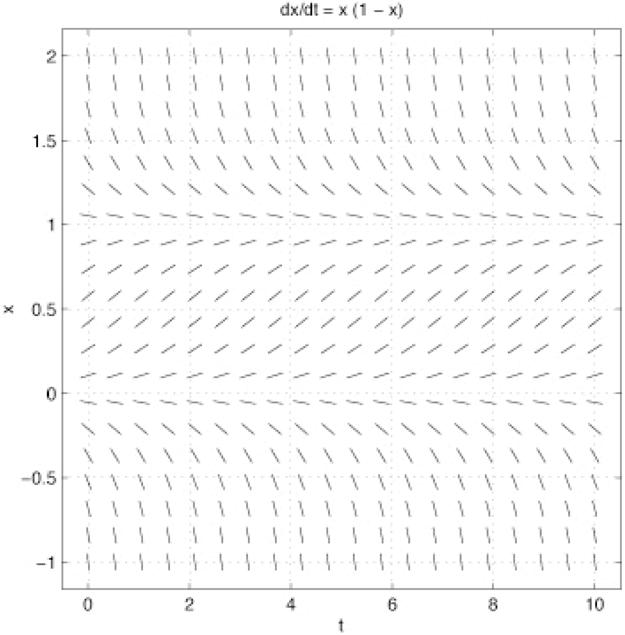
The phase line

Recall that the slope field for an autonomous differential equation, i.e., an equation of the form

$$\frac{dy}{dt} = f(y)$$

has a special property – slope marks are parallel along horizontal lines. In this case, there is clearly some redundancy in the slope field information.

We can replace the slope field by a **phase line**, which summarises the information in the slope field.


dx/dt = x (1 − x)
x
2
1.5
1
0.5
0
−0.5
−1
0
2
4
6
8
10
t

To draw a phase line for

$$\frac{dy}{dt} = f(y)$$

- Draw a y-line (y-axis).
- Find equilibrium solutions of the DE and mark them on the line.
- Find intervals of y for which $f(y) > 0$ (solutions started at such y values will increase as t increases). Draw upward pointing arrows on the y-line in these intervals.
- Find intervals of y for which $f(y) < 0$ (solutions started at such y values will decrease as t increases). Draw downward pointing arrows on the y-line in these intervals.

Example 1: Sketch the phase line for the DE

$$\frac{dy}{dt} = (y+2)(1-y)$$

Describe the long term behaviour of solutions.

Example 2: Sketch the phase line for the DE

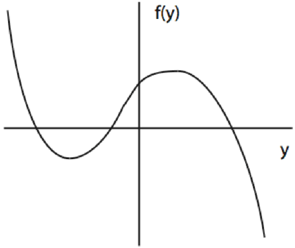
$$\frac{dy}{dt} = y^2(y+1)$$

Describe the long term behaviour of solutions.

Example 3: For the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ has the graph shown below, sketch the phase line and describe the long term behaviour of solutions.



It is possible to sketch solutions to a DE just from the phase line.

Example 4: Sketch solutions that might come from the same DE as the phase line shown below.

Note that a phase line contains information about whether solutions are increasing or decreasing as time increases but gives no information about the *speed* with which solutions are increasing or decreasing.

As a result, we cannot reconstruct solution curves *quantitatively* from a phase line, i.e., we get no information about the time at which a particular solution takes on a certain value.

In the previous example, we saw that for a given phase line there were different possible shapes for the solution curves plotted in the y-t plane. All the possibilities are qualitatively the same as each other but can be different in the (quantitative) slopes of the solution curves.

Long term behaviour of solutions

In cases where the Uniqueness Theorem applies, a solution that tends to an equilibrium point does not reach the equilibrium point in finite time. We write

$$y(t) \to y_0 \quad \text{as} \quad t \to \infty \quad (\text{or as } t \to -\infty).$$

In contrast, a solution that tends to $+\infty$ or $-\infty$ may reach $\pm\infty$ in finite time, or may never reach $\pm\infty$. We cannot tell which case we have from the phase line alone.

Example 5:

$$\frac{dy}{dt} = 1$$

Example 6:

$$\frac{dy}{dt} = 1 + y^2$$

These examples show that we cannot write

$$y(t) \to \pm\infty \text{ as } \ t \to \infty \text{ (or as } t \to -\infty)$$

based on evidence from the phase line alone - we would need more information about the actual solutions before making such a statement.

Instead, based on phase lines, we make statements like

$y(t) \to \infty$ as t increases

or

$y(t) \to \infty$ as t decreases.

Important ideas from today:

For an autonomous differential equation

$$\frac{dy}{dt} = f(y)$$

it can be useful to sketch the phase line.

The phase line contains information about equilibrium solutions, and shows whether other solutions are increasing or decreasing.

Information about the *speed* with which solutions are changing is lost in a phase line.