Department of Mathematics

# Maths 260 Assignment 4 Solutions

October 29, 2009

[ 84 marks in total ]

1. (6 marks total)

The matrix has repeated eigenvalue 2 with a single eigenvector $(1,0)^T$. We need to find the generalised eigenvector:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

One possible generalised eigenvector is $(0,1)^T$.

Hence the general solution is

$$\mathbf{Y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} + c_2 \left( t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \mathrm{e}^{2t} = \mathrm{e}^{2t} \begin{pmatrix} c_1 + tc_2 \\ c_2 \end{pmatrix}$$

The solution with $\mathbf{Y}(0) = (1,1)$ has $c_1 = c_2 = 1$, so

$$\mathbf{Y} = \mathrm{e}^{2t} \begin{pmatrix} 1+t \\ 1 \end{pmatrix}$$

2. (25 marks total)

Define the matrix in the question to be $\mathbf{A}$, that is,

$$\mathbf{A} = \begin{pmatrix} a & -2 \\ 1 & 0 \end{pmatrix}$$

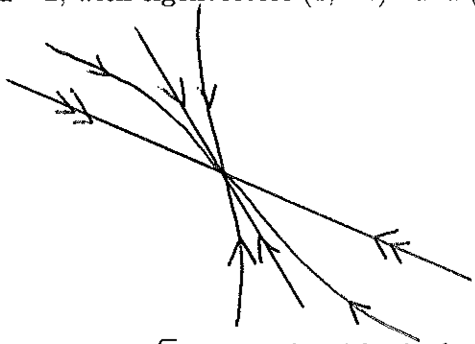
The eigenvalues of $\mathbf{A}$ are $\lambda = \dfrac{a \pm \sqrt{a^2-8}}{2}$.

We also have that trace$\mathbf{A} = a$ and $\det \mathbf{A} = 2$. Since $2 > 0$, the origin is never a saddle. If $a^2 < 8$ then the eigenvalues are complex and so the origin is a spiral. Otherwise, the eigenvalues are real, and since the origin is never a saddle, it must be a nodal source or sink. If $a > 0$ then the origin is a source, and if $a < 0$ then the origin is a sink. So to summarise: If $a < -2\sqrt{2}$, the origin is a nodal sink. If $a = -2\sqrt{2}$, the origin is an improper sink. If $-2\sqrt{2} < a < 0$ the origin is a spiral sink. If $a = 0$, the origin is a center. If $0 < a < 2\sqrt{2}$ the origin is a spiral source. If $a = 2\sqrt{2}$, the origin is an improper source. If $a > 2\sqrt{2}$, the origin is a nodal source.

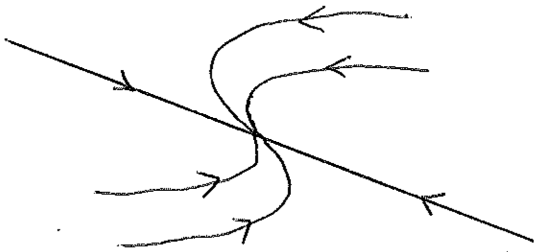
Note that $\mathbf{A}(0,1)^T = (-2,0)^T$, regardless of the value of $a$, so any spirals or improper nodes will be anticlockwise.

Summary and sketches, including boundary cases:

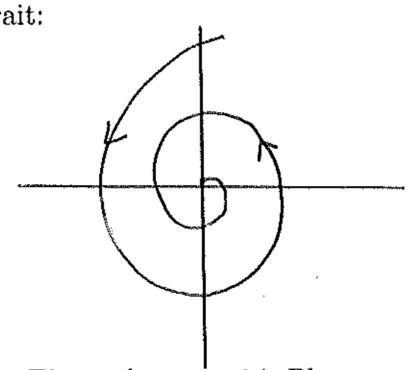
- $a < -2\sqrt{2}$: Origin is a nodal sink. Representative value of $a = -3$. Then the eigenvalues are $-1$ and $-2$, with eigenvectors $(1,-1)^T$ and $(-2,1)^T$. Phase portrait:



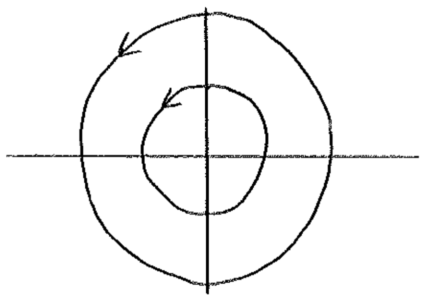
- $a = -2\sqrt{2}$. Eigenvalues are $-\sqrt{2}$ repeated, with single eigenvectors $(-\sqrt{2},1)^T$. Phase portrait:



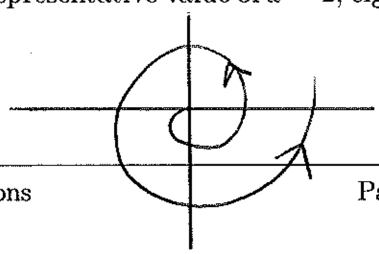
- $-2\sqrt{2} < a < 0$. Origin is a spiral sink. Representative value of $a = -2$. Eigenvalues are $-1 \pm i$. Phase portrait:



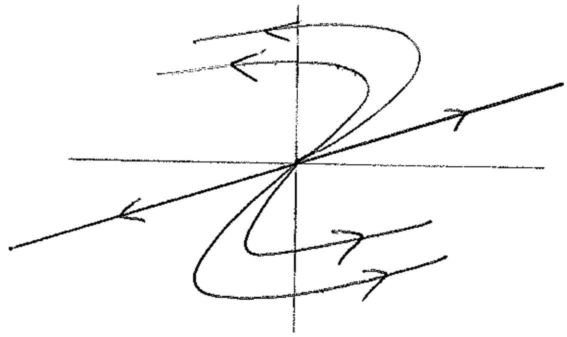
- $a = 0$. Origin is a center. Eigenvalues are $\pm i$. Phase portrait:



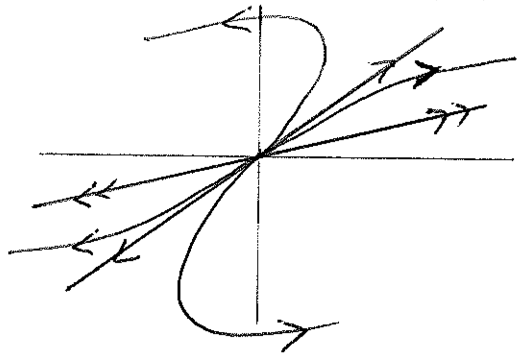
- $0 < a < 2\sqrt{2}$. The origin is a spiral source. Representative value of $a = 2$, eigenvalues are $1 \pm i$. Phase portrait:



- $a = 2\sqrt{2}$. Eigenvalues are $\sqrt{2}$ repeated, with single eigenvectors $(\sqrt{2}, 1)^T$. Phase portrait:



- $a > 2\sqrt{2}$: Origin is a nodal source. Representative value of $a = 3$. Then the eigenvalues are 1 and 2, with eigenvectors $(1,1)^T$ and $(2,1)^T$. Phase portrait:



3. (36 marks total)

   (a) Find the equilibria by solving $\dot{x} = 0 : x = 0$ or $y = 2x$, then $\dot{y} = 0 : y = 0$ or $x = \pm 2$. So the equilibria are $(0,0)$, $(2,4)$ and $(-2,-4)$.

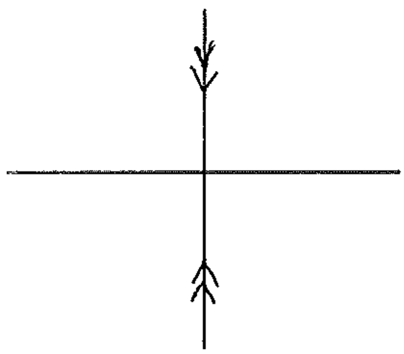
   - Find the Jacobian:

   $$J = \begin{pmatrix} 4x - y & -x \\ 2xy & x^2 - 4 \end{pmatrix}$$

   - For the equilibrium at $(0,0)$,

   $$J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$$

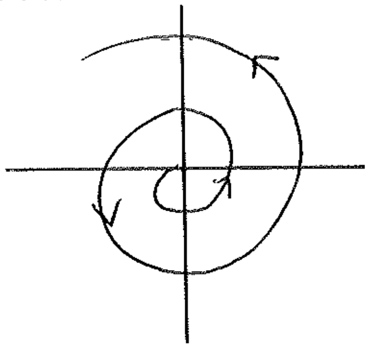
   which has eigenvalues $0, -4$ and eigenvectors $(1,0)^T$ and $(0,1)^T$ .
   As there is a zero eigenvalue, we cannot predict the behaviour in the nonlinear system based on linearisation alone.
   We need to look at the phase portrait of the full nonlinear system.



- For the equilibrium at $(2,4)$,

$$J(2,4) = \begin{pmatrix} 4 & -2 \\ 16 & 0 \end{pmatrix}$$

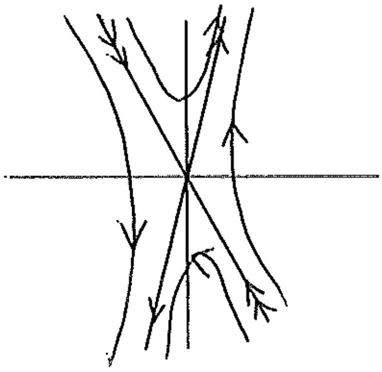
which has eigenvalues $2 \pm 5.2915i$ , thus $(2,4)$ is a spiral source . Picking the point $(0,1)^T$ and using the Jacobian, we find the direction vector $(-2,0)^T$. Therefore the spiral has a counter-clockwise direction.



- For the equilibrium at $(-2,-4)$,

$$J(-2,-4) = \begin{pmatrix} -4 & 2 \\ 16 & 0 \end{pmatrix}$$

which has eigenvalues $-8, 4$ and eigenvectors $(-1,2)^T$ and $(-1,-4)^T$ . So $(-2,-4)$ is a saddle .

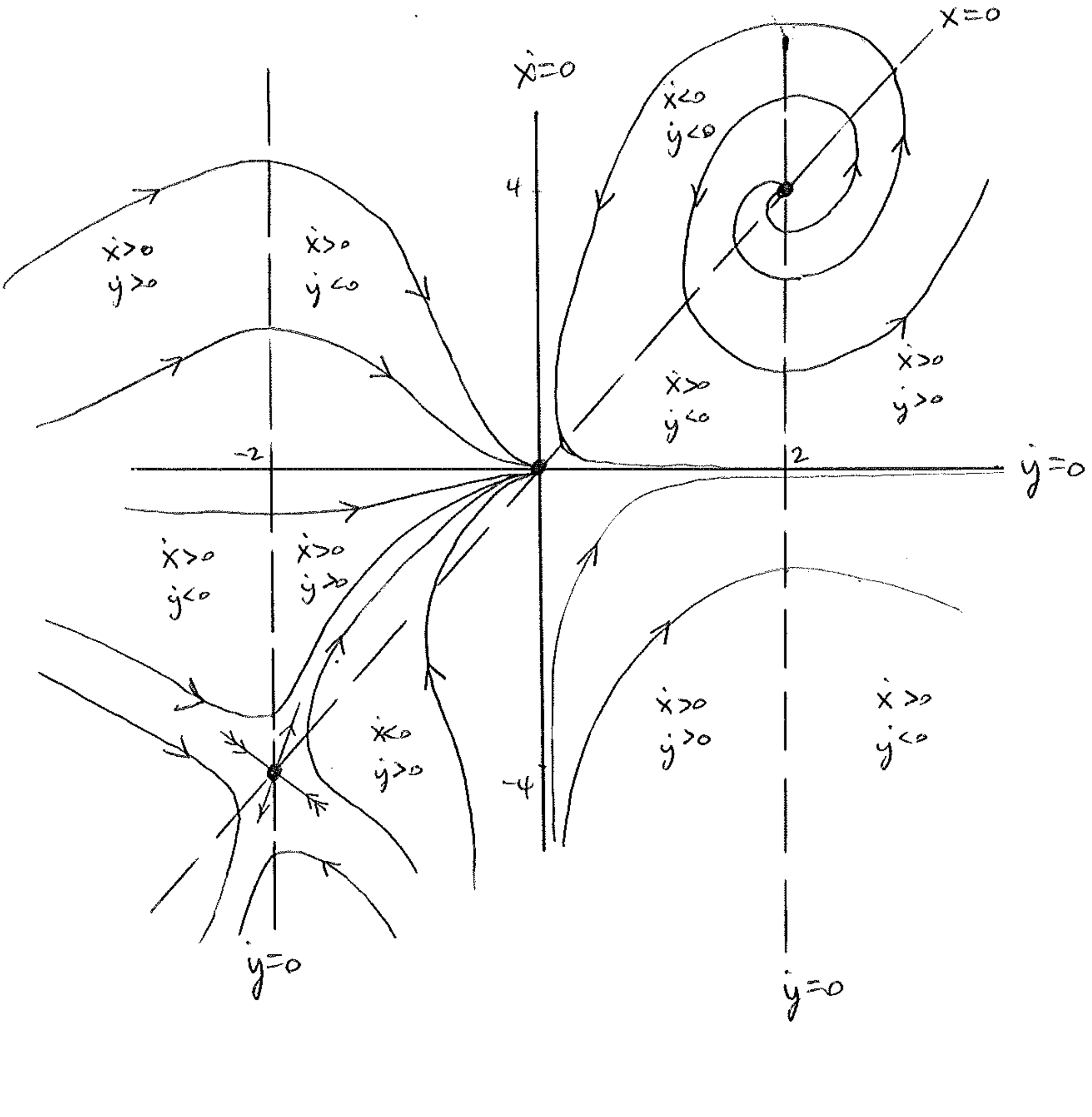


(b) Nullclines :

- The $x$–nullclines are $x = 0$ and $y = 2x$
- The $y$–nullcline are $y = 0$ and $x = \pm 2$ .
- $\dot{x} > 0$: $x > 0$ and $y < 2x$; or $x < 0$ and $y > 2x$ .
- $\dot{x} < 0$: $x < 0$ and $y < 2x$; or $x > 0$ and $y > 2x$ .
- $\dot{y} > 0$: $y > 0$ and $x < -2$ and $x > 2$; or $y < 0$ and $-2 < x < 2$ .
- $\dot{y} < 0$: $y < 0$ and $x < -2$ and $x > 2$; or $y > 0$ and $-2 < x < 2$ .

(c) Sketch of the phase portrait

The equilibrium at (0,0) is a degenerate node.



4. (10 marks total)

(a) (

i. The term $-aSI$ in the equation for $dS/dt$ is the term modelling the negative effect on the $S$ population of interactions between susceptible people with infectious people, ie. susceptible people become infectious.

ii. The term $aSI$ in the equation for $dI/dt$ is the term modelling the positive effect on the $I$ populations of interactions between susceptible people with infectious people, ie. susceptible people become infectious.

iii. The term $-bI$ in the equation for $dI/dt$ is the term modelling infectious people recovering which is a negative effect upon the infectious population, ie. infected people recover.

iv. The term $bI$ in the equation for $dR/dt$ is the term modelling the positive effect upon the $R$ population of infectious people recovering, ie. infected people recover.

(b) Include a new term in $\frac{dS}{dt}$ to model vaccination which removes people from the susceptible population. Include a new term in $\frac{dR}{dt}$ to model an increase in the recovered/immune population to show that vaccination has increased the number of immune population. These two equations are now:

$$\frac{dS}{dt} = -aSI - cS$$
$$\frac{dR}{dt} = bI + cS$$

where $c$ is the rate at which susceptibles are vaccinated and the rate at which susceptibles become immune.

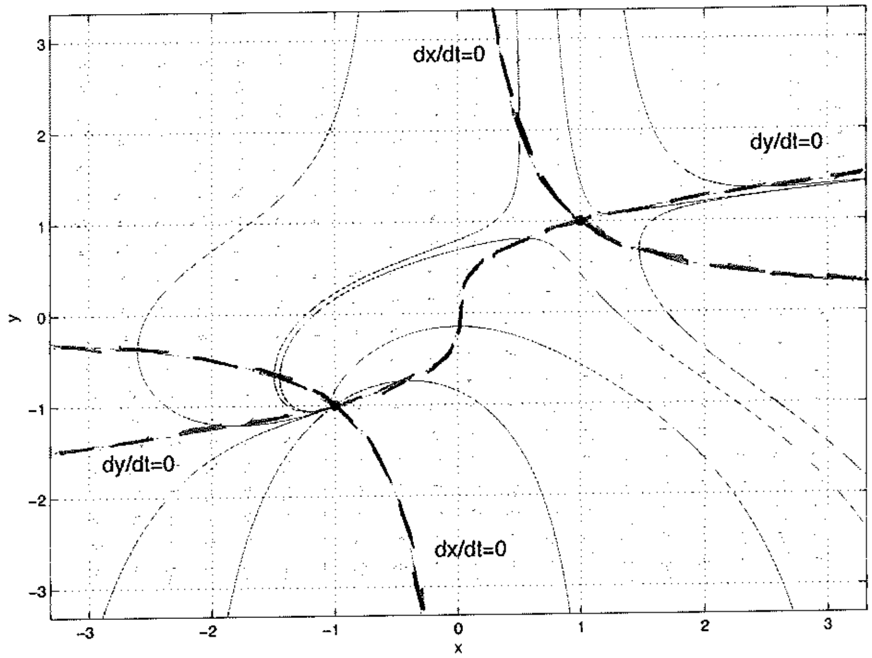
(c) First find all equilibria of the system by finding all points $(S, I, R)$ such that $\frac{dS}{dt} = 0, \frac{dI}{dt} = 0, \frac{dR}{dt} = 0$.

The system is a nonlinear system so then linearise by finding the Jacobian. Classify the equilibria by finding the eigenvalues and eigenvectors for the Jacobian for each equilibria. If any have zero or repeated eigenvalues, then this may be insufficient to sketch the full phase portrait.

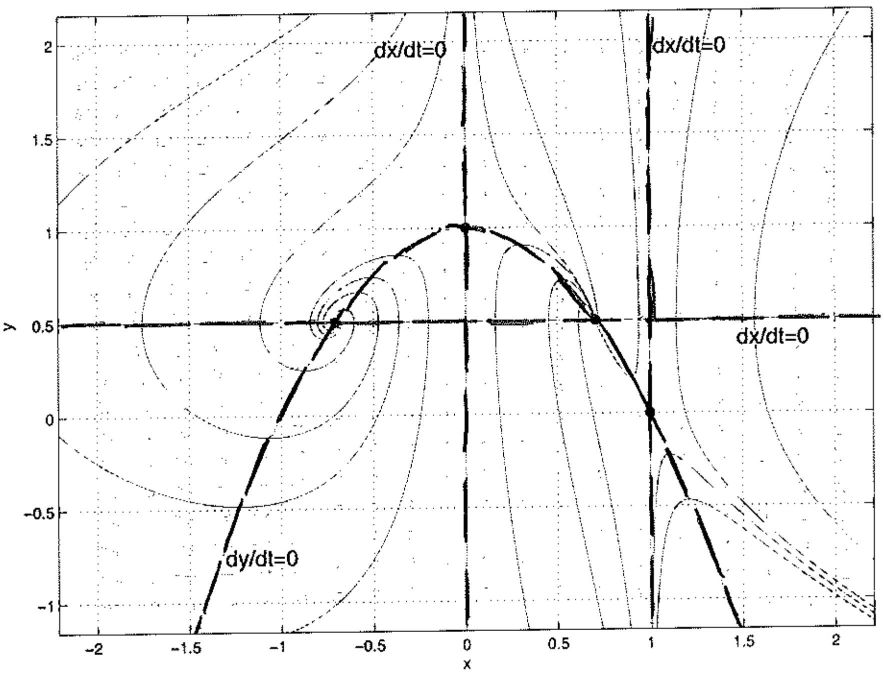
Find all the nullclines, ie. where $dS/dt = 0$, $dI/dt = 0$ and $dR/dt = 0$, to help to sketch the full phase portrait. By calculating the regions where $\dot{S}, \dot{I}, \dot{R}$ are greater than zero and calculating the regions where $\dot{S}, \dot{I}, \dot{R}$ are less than zero, it is possible to draw representative solutions and show the longterm behaviour of solutions in the phase plane.

The system of equations can be integrated numerically using, for example, Matlab. This will give approximate solutions for some particular choice of coefficients. By varying the coefficients it is then possible to investigate the effect of different infection rates and see how solutions change.

5. (7 marks total)




(a)




(b)