Maths 260 Midsemester Test 2007S

Some Answers (much fuller than required).

1. (7 marks) Consider the differential equation

$$\frac{dy}{dt} + \frac{y}{t} = 3.$$

(a) Show that
$$y_p(t) = \frac{1}{t} + \frac{3t}{2}$$
is a solution to the differential equation.

(b) Find the general solution to the differential equation.

(c) Find a solution to the differential equation that satisfies the initial condition $y(1) = -1$.

(a) Substitute y_p into the left side of the DE and also into the right side. If these two quantities are equal then y_p is a solution to the DE:

LHS: $$\frac{dy_p}{dt} + \frac{y_p}{t} = \left(-\frac{1}{t^2} + \frac{3}{2}\right) + \frac{1}{t}\left(\frac{1}{t} + \frac{3t}{2}\right)$$
$$= -\frac{1}{t^2} + \frac{3}{2} + \frac{1}{t^2} + \frac{3}{2}$$
$$= 3 = \text{RHS}$$

so y_p is a solution.

(Note that the soln is not defined for $t>0$ and nor is the differential equation. So, the expression for y_p actually defines 2 solns, one for $t>0$ and one for $t<0$.

(b) There are 2 methods for solving this equation.

Method A: The DE is linear with $a(t) = \frac{1}{t}$

Calculate the integrating factor:
$$\mu(t) = \exp\left(\int a(t)\,dt\right) = \exp\left(\int \frac{1}{t}\,dt\right)$$
$$= \exp\left(\ln|t|\right)$$

There are 2 cases (although full marks were obtained by considering just one case): ②

① $t>0 \Rightarrow \mu = t$

Then $t\frac{dy}{dt} + t\frac{y}{t} = 3t$

or $\frac{d}{dt}(ty) = 3t$

$\Rightarrow ty = \int 3t\,dt$

$= \frac{3t^2}{2} + c$

$\Rightarrow y = \frac{3t}{2} + \frac{c}{t}$ for c an arbitrary constant

② $t<0 \Rightarrow \mu = -t$

Then $-t\frac{dy}{dt} - \frac{ty}{t} = -3t$

which reduces to the same equation as in case ① and so the general solution is the same.

<u>method B</u> First solve the homogeneous equation

$\frac{dy}{dt} = \frac{-y}{t}$. This is separable.

$\int \frac{dy}{y} = \int -\frac{1}{t}\,dt \qquad y \neq 0$

$\ln|y| = -\ln|t| + k = \ln\left(\frac{1}{|t|}\right) + k$

$|y| = e^k e^{\ln\left(\frac{1}{|t|}\right)}$

$y = \pm e^k e^{\ln\left(\frac{1}{|t|}\right)}$

$= \frac{c}{t}$ if $t>0$ or $\frac{\tilde{c}}{t}$ if $t<0$

In these expressions, c and $\tilde{c}$ are arbitrary non-zero constants. Note that $y=0$ is also a solution, and so we get the general solution to the homogeneous equation being $y(t) = \frac{d}{t}$ for d an arbitrary constant. The same formula works for both $t>0$ and for $t<0$

To get the general solution to the nonhomogeneous equation, we just add y_p and $y(t) = \frac{d}{t}$

ie. the general solution is

$$y(t) = \frac{d}{t} + \frac{1}{t} + \frac{3t}{2}$$

$$= \frac{\tilde{d}}{t} + \frac{3t}{2}$$ for $\hat{d} = d+1$ an arbitrary constant.

2. (16 marks) Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = \mu y + 4y^3.$$

(a) For the case $\mu = 1$, find all equilibrium solutions and determine their type (e.g., sink, source). Sketch the phase line.

(b) Repeat (a) for the case $\mu = 0$.

(c) Repeat (a) for the case $\mu = -1$.

(d) Now let μ vary. Locate the equilibrium solutions and determine their type. Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram.

(a) $\frac{dy}{dt} = y + 4y^3$

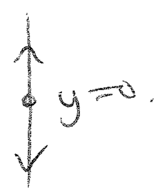
Equilibrium solutions satisfy $y + 4y^3 = 0$, which has one real soln $y=0$. We are not interested in complex-valued solutions.

Writing $f(y) = y + 4y^3$, we look at the sign of $\frac{\partial f}{\partial y}$ to determine the type of the equilibrium at $y=0$

$$\frac{\partial f}{\partial y} = 1 + 12y^2 = 1 \quad \text{if } y = 0$$

Since this quantity is positive, $y=0$ is a source.

Phase portrait:



(b) $\frac{dy}{dt} = 4y^3$.

The only equilibrium is $y=0$

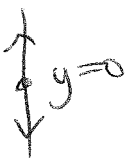
$$\left.\frac{\partial f}{\partial y}\right|_{y=0} = \left.12y^2\right|_{y=0} = 0$$

so linearisation is unhelpful for determining the type of the equilibrium.

However, $\frac{dy}{dt} = 4y^3 \Rightarrow \frac{dy}{dt} > 0$ if $y > 0$

and $\frac{dy}{dt} < 0$ if $y < 0$.

So the phase portrait is



and the equilibrium is a source.

(c) $\frac{dy}{dt} = -y + 4y^3$

This has equilibria when $-y(1-4y^2) = 0$

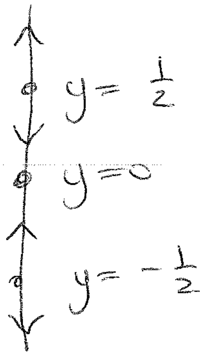
ie. $y=0$ or $y = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$\left.\frac{\partial f}{\partial y}\right|_{y=0} = \left.-1+12y^2\right|_{y=0} = -1$ so $y=0$ is a sink.

$\left.\frac{\partial f}{\partial y}\right|_{y=\frac{1}{2}} = -1 + 12\left(+\frac{1}{2}\right)^2 = 2$ so $y = \frac{1}{2}$ is a source.

$\left.\frac{\partial f}{\partial y}\right|_{y=-\frac{1}{2}} = -1 + 12\left(-\frac{1}{2}\right)^2 = 2$ so $y = -\frac{1}{2}$ is a source.

The phase portrait is



(d) $\dfrac{dy}{dt} = \mu y + 4y^3 = 0$ if $y=0$ or $y = \pm\sqrt{\dfrac{-\mu}{4}}$

Thus there is one equilibrium if $\mu \geq 0$ and three equilibria if $\mu < 0$. There is a bifurcation at $\mu = 0$.

$$\frac{\partial f}{\partial y} = \mu + 12y^2$$

so $\left.\dfrac{\partial f}{\partial y}\right|_{y=0} = \mu \quad \Rightarrow$ $y=0$ is a source if $\mu > 0$ and a sink if $\mu < 0$

$$\left.\frac{\partial f}{\partial y}\right|_{y=\pm\sqrt{\frac{-\mu}{4}}} = \mu + 12\left(\sqrt{\frac{-\mu}{4}}\right)^2 = \mu - 3\mu = -2\mu$$

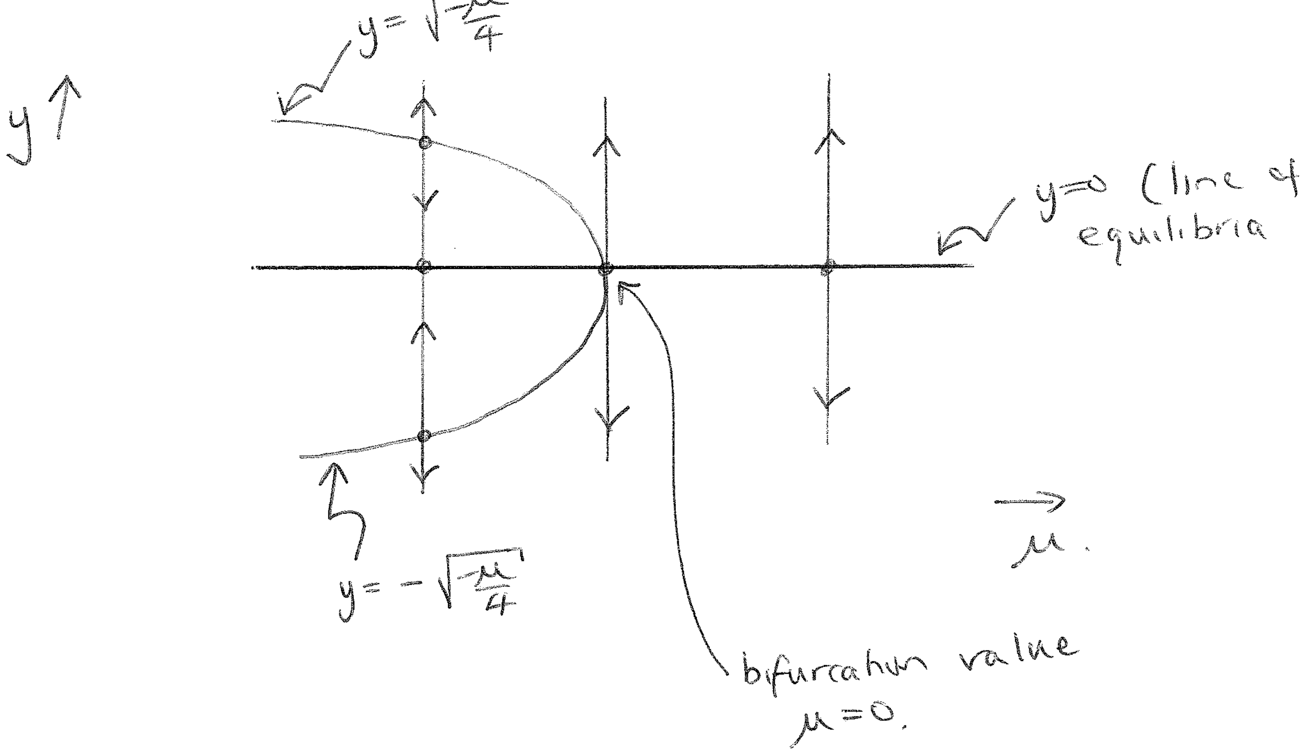
so $y = \pm\sqrt{\dfrac{-\mu}{4}}$ is a source for $\mu < 0$.

(and there is no solution of this form for $\mu > 0$)

The stability of the equilibrium for the case $\mu = 0$ is given in (b) above.

(d) continued.

Bifurcation diagram is, therefore:



3. (9 marks) Consider the system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find the general solution to the system of equations.

(b) Sketch the phase portrait corresponding to these equations. Include in your picture the solution that satisfies the initial condition $x(0) = 1,\ y(0) = 1$ and also the solution that satisfies $x(2) = -1,\ y(2) = 1$.

(a) The eigenvalues of the matrix are -1 and -2 (matrix is triangular, so e-values are on the diagonal).

If $\lambda = -1$, $(A - \lambda I)\underset{\sim}{v} = \begin{pmatrix} -1-(-1) & 3 \\ 0 & -2-(-1) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \begin{pmatrix} 0 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ if } y = 0$$

so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an e-vector for $\lambda = -1$

If $\lambda = -2$, $(A - \lambda I)\underset{\sim}{v} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if

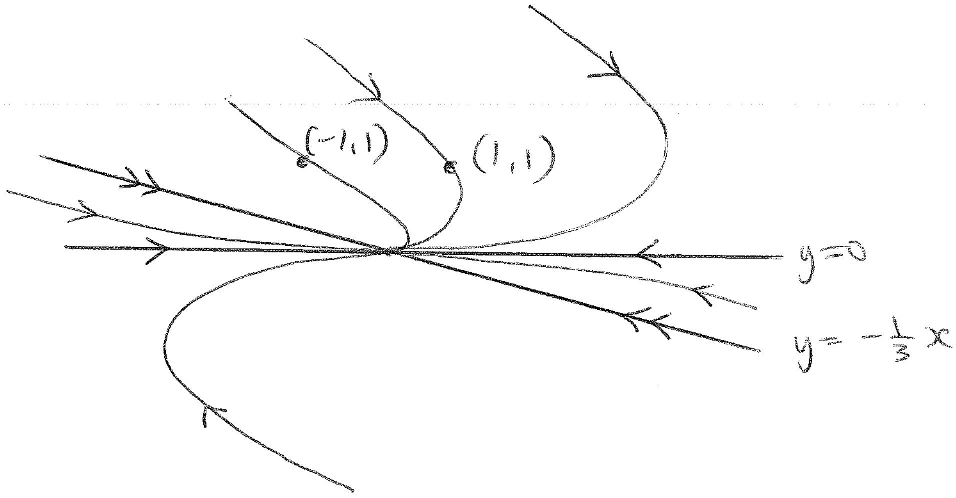
$x = -3y$

$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an e-vector for $\lambda = -2$

So the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(b)



For full marks, you needed to

(i) correctly draw the straight line solns

(ii) show that all solns other than those with $y=-\frac{1}{3}x$ are tangent to the x-axis

(iii) correctly draw the soln through $(1,1)$ and $(-1,1)$

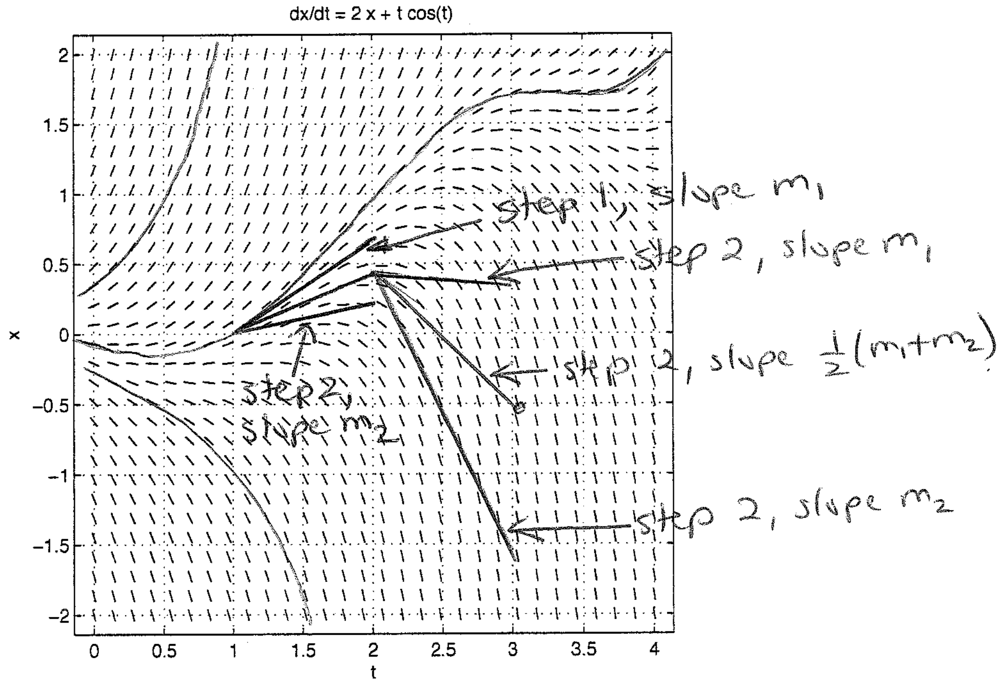
(iv) overall, have a nice picture with one or more solutions in each of the 4 regions separated by the straightline solutions.

10

4. (13 marks) Consider the differential equation

$$\frac{dy}{dt} = 2y + t\cos t$$

with initial condition $y(1) = 0$. The slope field for the differential equation is shown below. (Spare copies of the slope field are available if you muddle up your answer on the first attempt.)



(a) On the picture above sketch by hand, as accurately as you can, three representative solutions to the differential equation, including the solution that satisfies the initial condition given above.

(b) On the same printout, draw by hand (and ruler) the graph you would obtain if you used mid-point Euler's method, with stepsize $h = 1.0$, to approximate the solution to the initial value problem at $t = 3$. You do not need to perform any calculations with mid-point Euler's method to answer this part of the question - just use the slope marks shown on the picture.

(c) Estimate the error in your mid-point Euler approximation, using the solution you sketched in (a). What would you expect to happen to the error if you used mid-point Euler with $h = 0.5$ to approximate the solution to the initial value problem at $t = 3$?

(d) Find all points (t_0, y_0) in the $t-y$ plane for which there is a unique solution to the initial value problem consisting of the differential equation plus the initial condition $y(t_0) = y_0$. Given a reason for your answer.

(a) 3 representative solutions are shown on the previous page. Others are possible. Note that you should not draw 3 solutions passing through the same initial condition.

(b) Note the typo in this question. The method required was Improved Euler, not midpoint Euler. To ensure no one was harmed by this error, some marks were given for almost any sensible answer and the test was marked out of 40, not 45. This will be discussed further in lectures.

The picture on the previous page shows the solution for Improved Euler.

(Note that many possible soln curves in (a) or numerical solns in (b) are "correct", since it is impossible to be completely accurate with the information given. The answer to (c) below is for the solns sketched on the previous page.)

(c) From (a), $y(3) \approx 1.7$ for the solution through $t=1$, $y=0$.

From (b), $y(3) \approx -0.5$ for Improved Euler.

Hence, error $\approx |1.7 - (-0.5)| = 2.2$.

If a smaller stepsize is used, the error should decrease. Since Improved Euler is order 2, we might expect the error to decrease by a factor of $2^2 = 4$ (although, since $h=1$ is such a large stepsize, this theoretical factor could be very inaccurate. It is in the limit of small h that we expect a halving of stepsize to lead to a decrease in error by a factor of 4.).

(d) $f(t,y) = 2y + t\cos t$

is a continuous function of t and y for all t and y. Similarly,

$\frac{\partial f}{\partial y} = 2$ is a continuous function of t and y for all t and y.

Hence, by the Existence and Uniqueness Theorem, we expect the initial value problem

$$\frac{dy}{dt} = 2y + t\cos t, \quad y(t_0) = y_0$$

to have a unique solution for all t_0, y_0.

(Note: the question did not ask about only the initial condition $y(1) = 0$).