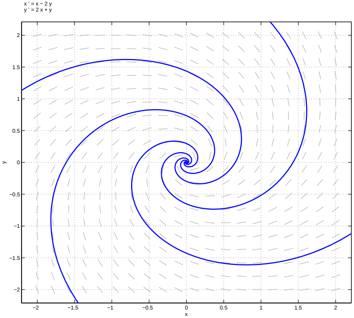
Maths 260 Lecture 22

- **Topic for today:** Linear systems with complex eigenvalues
- **Reading for this lecture:** BDH Section 3.4
- **Suggested exercises:** BDH Section 3.4; 1, 3, 5, 7, 9, 11, 23
- **Reading for next lecture:** BDH Section 3.5

Example 1

- Here is the slope field and some solutions for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$$



- Why does the phase portrait have no straight-line solutions?

- Calculate the eigenvalues:

- In this example the eigenvalues are complex.
- We saw in earlier lectures that straight-line solutions result from real eigenvalues.
- We found that

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

  is always a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

  if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{v}$.
- However, the corresponding solution curve will not be a straight line if λ is not real.

Example 1 again:

- Find two linearly independent solutions of the system:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$$

- How do we interpret a complex-valued solution? We would like a real-valued solution.

Theorem:

- Consider the system
$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$
- If $\mathbf{Y}(t)$ is a complex-valued solution to the system, write
$$\mathbf{Y}(t) = \mathbf{Y_R}(t) + i\mathbf{Y_I}(t)$$
where $\mathbf{Y_R}(t)$ and $\mathbf{Y_I}(t)$ are real-valued functions.
- Then $\mathbf{Y_R}(t)$ and $\mathbf{Y_I}(t)$ are solutions to the system and are linearly independent.

Proof:

Example 1 again:

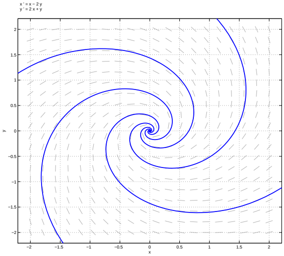
- Find two linearly independent real-valued solutions of the system:

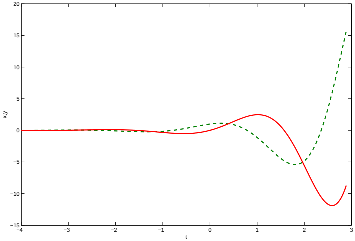
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$$

  and hence write down the general solution in terms of real-valued functions.

Phase portrait

- We see from the general solution that each component of $\mathbf{Y}(t)$ oscillates between positive and negative values and that the amplitude of each component grows exponentially.
- Phase portrait and components of solution with $x(0) = 1$, $y(0) = 0$.





- In Example 1, we found two linearly independent real-valued solutions by taking the real and imaginary parts of the complex-valued solution

$$\mathrm{e}^{(1+2i)t}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

- What if we instead used the real and imaginary parts of the other complex-valued solution we found, i.e.

$$\mathrm{e}^{(1-2i)t}\begin{pmatrix} -i \\ 1 \end{pmatrix}?$$

- The other complex-valued solution also gives us two real-valued solutions.
- These solutions are just multiples of the real-valued solutions already found.
- Thus, using the other complex-valued solution gives no new information
- We can form the general solution using the real and imaginary parts of just one of the complex conjugate pair of solutions.

In general...

- Suppose **A** has complex eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.
- Then the system
$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$
has a solution of the form
$$\mathbf{Y}(t) = \mathrm{e}^{(\alpha+i\beta)t}\mathbf{Y_1}$$
where $\mathbf{Y}_1$ is the eigenvector corresponding to eigenvalue λ_1.
- Expanding the exponential yields
$$\mathbf{Y}(t) = \mathrm{e}^{(\alpha+i\beta)t}\mathbf{Y_1} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))\mathbf{Y_1}$$
- So the general solution is a combination of exponential and trigonometrical terms. The qualitative behaviour of solutions depends on α and β.

- Notice that the 2 by 2 matrix

$$\mathbf{A} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

has eigenvalues $\alpha \pm i\beta$. (**Exercise:** check this!)

- Using pplane we can see how varying α and β in the equation

$$\dot{x} = \alpha x - \beta y$$
$$\dot{y} = \beta x + \alpha y$$

changes the phase portrait.

Spiral Source

- **Case 1:** If $\alpha > 0$, then $\exp(\alpha t) \to \infty$ as $t \to \infty$ so solution curves spiral away from the origin, and the equilibrium at the origin is called a spiral source.

Typical phase portraits for a spiral source:

Sprial Sink

- **Case 2:** If $\alpha < 0$, then $\exp(\alpha t) \to 0$ as $t \to \infty$ so solution curves spiral into the origin, and the equilibrium at the origin is called a spiral sink.

Typical phase portraits for a spiral sink:

Centre

- **Case 3:** If $\alpha = 0$, then $\exp(\alpha t) = 1$ for all t and solutions are periodic; solution curves return to their initial point in the phase plane and retrace the same curve over and over again. In this case, the equilibrium at the origin is called a centre.

Typical phase portraits for a centre:

Example 1 again:

- Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \mathbf{Y}$$

- The eigenvalues are $1 \pm 2i$, i.e. $\alpha = 1$, $\beta = 2$, and so the origin is a spiral source.

Which direction?

- To determine whether spiral is clockwise or anticlockwise, evaluate the vector field at a point.
- For example, at $(x, y) = (0, 1)$ on the y-axis, the direction of the solution through this point is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

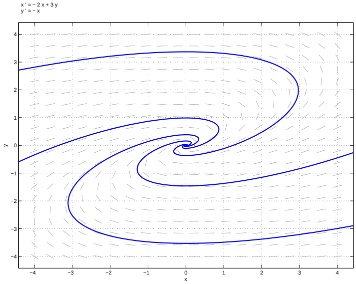
  which is a vector pointing up and left.
- This is not consistent with a clockwise spiral so solutions must spiral around the origin in an anticlockwise direction.

Example 2

- Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{Y}$$

- Direction field and some solutions:



Exercise: Show that the general solution to the system, written in terms of real-valued functions, is

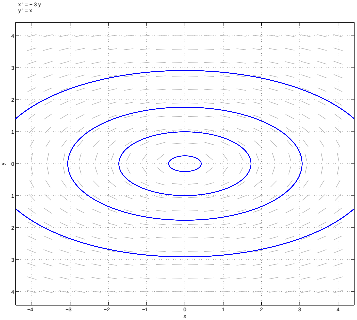
$$\mathbf{Y}(t) = c_1 \mathrm{e}^{-t} \begin{pmatrix} \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t \end{pmatrix} + c_2 \mathrm{e}^{-t} \begin{pmatrix} \sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t \end{pmatrix}$$

Example 3

- Sketch the phase portrait for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \mathbf{Y}$$

- Direction field and some solutions:



Exercise: Show that the general solution to the system, written in terms of real-valued functions, is

$$\mathbf{Y}(t) = c_1 \begin{pmatrix} 3\cos\sqrt{3}t \\ \sqrt{3}\sin\sqrt{3}t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin\sqrt{2}t \\ -\sqrt{3}\cos\sqrt{3}t \end{pmatrix}$$

Example 4

- Find the general solution, expressed in terms of real-valued functions, for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{Y}$$

- Determine the long term behaviour of solutions.

Important ideas from today

- If $\mathbf{A}$ is a matrix with an eigenvalue $\lambda = \alpha + i\beta$ and corresponding eigenvector $\mathbf{v}$, then

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{v}$$

is a solution to

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

regardless of whether λ is real or complex.

- However, the corresponding solution curve will not be a straight line if λ is complex.

If **A** is a 2 by 2 matrix with eigenvalue $\lambda = \alpha + i\beta$ with $\beta \neq 0$, there are three possibilities:

1. If $\alpha > 0$, the origin is a spiral source and solutions spiral away from the origin as t increases.
2. If $\alpha < 0$, the origin is a spiral sink and solutions spiral towards the origin as t increases.
3. If $\alpha = 0$, the origin is a centre and solutions are periodic, forming closed curves around the origin.

We determine the direction in which solutions spiral (i.e. clockwise or anticlockwise) by examining the direction of the solution through one point near the origin.