THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2008
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: TWO hours)

NOTE: Answer **ALL** questions. Show **ALL** your working. 100 marks in total.

1. (15 marks)

(a) Find a solution to the initial value problem

$$\frac{dy}{dt} = -y - t, \quad y(0) = 2.$$

(b) (i) Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 1 - t.$$

(ii) Describe the way solutions to this differential equation behave as t gets very large.

2. (16 marks) Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = \mu y - y^3.$$

(a) For the case $\mu = -1$, find all equilibrium solutions and determine their type (e.g., sink, source). Sketch the phase line.

(b) Repeat (a) for the case $\mu = 0$.

(c) Repeat (a) for the case $\mu = 1$.

(d) Now let μ vary. Locate the equilibrium solutions and determine their type. Sketch the bifurcation diagram. Be sure to label the main features of the bifurcation diagram. Show all your working.

3. (17 marks) Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ a & 0 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a source.

(b) For the choice $a = -2$, find the general solution to the differential equation. Express your answer in terms of real-valued functions. Sketch the corresponding phase portrait.

(c) For the choice $a = -1$, find the general solution to the differential equation. Express your answer in terms of real-valued functions. Sketch the corresponding phase portrait.

4. (17 marks) This question is about the differential equation

$$\frac{dy}{dt} = \frac{y}{t} + 1, \quad t > 0.$$

(a) Show that

$$y(t) = t \ln t + kt$$

is a solution to the differential equation for $t > 0$ and for all choices of the real constant k. (In this expression, $\ln t$ is the natural logarithm of t.)

(b) Now consider the initial value problem consisting of the differential equation given above together with the initial condition $y(1) = 2$. Use two steps of Euler's method to calculate an approximate value of solution of the initial value problem at final time $t = 2$. Retain three decimal places of accuracy in your answer.

(c) For the initial value problem as in (b), use one step of Improved Euler's method to calculate an approximate value of the solution of the initial value problem at final time $t = 2$.

(d) Compute the error in the approximations to $y(2)$ that you calculated in (b) and (c). (Hint: use the solution given in (a) with an appropriate choice of the constant k. You may use the fact that $\ln 2 \approx 0.6931$.) Which approximation is more accurate? Is this what you expect? Give a reason for your answer.

(e) Use the Existence and Uniqueness Theorems to show that the initial value problem given in (b) has a unique solution.

5. (25 marks) Consider the following system of equations:

$$\frac{dx}{dt} = x(4 - 2x - y),$$
$$\frac{dy}{dt} = y(3 - x - y)), \quad x \geq 0, \; y \geq 0.$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find all equilibrium solutions and determine their type (e.g., spiral source, saddle). For each of TWO of the equilibria you find, draw a phase portrait showing the behaviour of solutions near that equilibrium.

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system. Your phase portrait should show the behaviour of solutions near the equilibria, and should show various solution curves *including* those passing through the following initial conditions:

(i) $(x(0), y(0)) = (1, 3)$;
(ii) $(x(0), y(0)) = (0, 2)$;
(iii) $(x(0), y(0)) = (1, 1)$.

Make sure you show clearly where solution curves go as $t \to \infty$.

6. (10 marks) The following equations model the growth of two populations living in a reserve:

$$\frac{dx}{dt} = x - \frac{x^2}{20} + 2xy,$$
$$\frac{dy}{dt} = 0.3y - \frac{y^2}{100} - 30xy, \quad x \geq 0, y \geq 0,$$

where x and y are measured in hundreds of animals (i.e., $x = 1$ means there are one hundred animals of type x) and t is measured in years.

(a) Explain briefly the physical significance of each term in the model.

(b) It is decided to try to reduce the size of population x by removing 200 animals of type x from the reserve in each year. Describe how you could modify the model to include this effect.

(c) Briefly describe some methods you could use to analyse these equations to get information about solutions to the model. You do not have to do any calculations to answer this part of the question.

Candidate's Name: ______________________________ ID No: _______________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 5