THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2005
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: THREE hours)

NOTE: Answer all questions. Show all your working. 100 marks in total.

1. (12 marks)

(a) Solve the initial value problem

$$\frac{dy}{dt} = y^2 t, \quad y(1) = -\frac{1}{2}.$$

(b) Find the general solution to

$$t\frac{dy}{dt} = 2y + t^3 e^{-2t}.$$

2. (12 marks) Consider the following differential equation:

$$\frac{dy}{dt} = a + 1 - y^2.$$

(a) Find all equilibrium solutions and determine their types (e.g., sink, node).

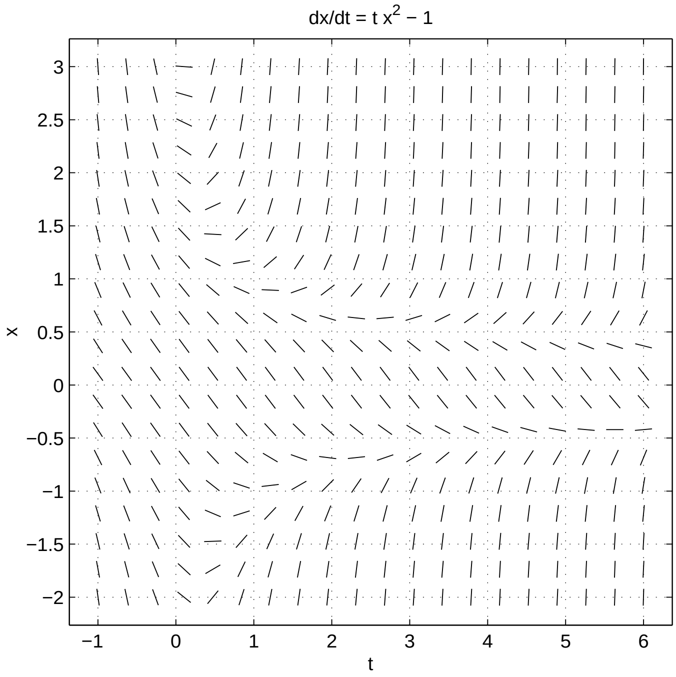
(b) Draw the bifurcation diagram. Identify any values of a for which a bifurcation exists.

(c) Describe the behaviour of solutions as t increases in the case $a = 3$.

3. (12 marks) Consider the differential equation

$$\frac{dx}{dt} = tx^2 - 1.$$

The slope field for this differential equation is shown below.



Another copy of the slope field is provided on the green answer sheet attached to the back of the question paper. Use this answer sheet to answer part (c) of this question.

(a) Very briefly describe how to draw a slope field by hand.

(b) Very briefly describe how to use a slope field to determine the behaviour of a solution to a differential equation.

(c) Use the slope field **on the special answer sheet provided** to sketch the solution to the initial value problem

$$\frac{dx}{dt} = tx^2 - 1, \quad x(1) = -1.$$

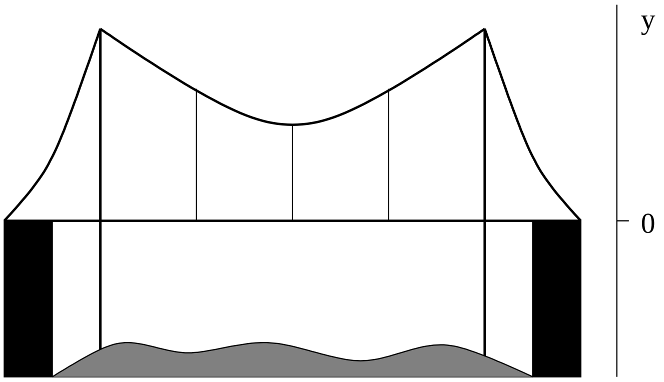
Hence estimate the value of this solution at $t = 3$. Hand in the answer sheet with your answer book.

(d) Use Euler's method with stepsize $h = 1$ to calculate an approximate value at $t = 3$ for the solution to the initial value problem in part (c).

(e) What do you expect to happen to the approximate value you found in (d) above if you used Euler's method with $h = 0.1$ instead? (You do not need to perform any extra calculations with Euler's method to answer this part of the question.)

(f) For what values of t_0 and x_0 will there be a unique solution to the differential equation satisfying the initial condition $x(t_0) = x_0$? Give a reason for your answer. You do not need to solve the differential equation to answer this question.

4. (8 marks) Consider the following second-order differential equation, which models the behaviour of a suspension bridge:



$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + h(y) = -g,$$

where $y(t)$ is the height of the bridge in metres with respect to the the position where the suspension cables are not stretched or compressed, $g = 10$ (gravity), $\alpha = 0.01$ (damping), and,

$$h(y) = \begin{cases} y, & \text{if } y < 0; \\ 2y, & \text{if } y \geq 0; \end{cases}$$

(a) Show that this equation can be rewritten as the following system of differential equations:

$$\begin{aligned} \frac{dy}{dt} &= z, \\ \frac{dz}{dt} &= -\alpha z - h(y) - g. \end{aligned}$$

(b) Show that $y = -10$, $z = 0$ is an equilibrium point for the system of equations. Give a physical explanation of why the y coordinate of this point is negative.

(c) Near this equilibrium solution, the system of equations given in part (a) above becomes:

$$\begin{aligned} \frac{dy}{dt} &= z, \\ \frac{dz}{dt} &= -\alpha z - y - g. \end{aligned}$$

Determine the type (e.g., saddle, spiral source) of the equilibrium point in the system. What do you expect to happen to the bridge if the initial y value is close to the equilibrium value?

(d) The bridge is to be tested by placing 100 heavy trucks filled with sand on the bridge. Briefly describe how you could modify the system of equations to model the effect of this?

5. (15 marks)

(a) Consider the system of equations

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix} \mathbf{Y}$$

(i) Determine the general solution. Express your answer in terms of real-valued functions.

(ii) Carefully sketch the phase portrait for the system.

(iii) Describe the long term behaviour of solutions to the system.

(b) Repeat steps (i)-(iii) for the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix} \mathbf{Y}.$$

6. (13 marks)

(a) Solve the following initial value problem:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = t + e^{-t}, \quad y(0) = \frac{1}{4}, \ \frac{dy}{dt}(0) = 0.$$

(b) Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

for all choices of the constants $p > 0$ and $q > 0$. Hence show that all solutions tend to zero as $t \to \infty$.

7. (18 marks) Use the grid provided on the green answer sheet attached to the back of the question paper for your answer to this question. Hand in the answer sheet with your answer book.

Consider the following system of equations:

$$\begin{aligned}\frac{dx}{dt} &= xy \\ \frac{dy}{dt} &= x^2 - y - 1\end{aligned}$$

(a) Show that there are three equilibrium solutions and determine their types (e.g., saddle, spiral sink).

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait for the system. Include in your phase portrait the solution curves passing through the initial conditions

(i) $(x(0), y(0)) = (0.5, 0)$;

(ii) $(x(0), y(0)) = (-1, 1)$.

Make sure you show clearly where these solution curves go as t increases and decreases.

8. (10 marks) In this question you may use the table of Laplace transforms attached.

(a) If

$$h(t) = \begin{cases} 0, & t < 3, \\ t - 3, & t \geq 3. \end{cases}$$

show that

$$\mathcal{L}\{h(t)\} = \frac{e^{-3s}}{s^2}.$$

(b) Show that

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+2)}\right\} = \mathcal{U}_3 e^{-2(t-3)}$$

where

$$\mathcal{U}_3 = \begin{cases} 0, & t < 3, \\ 1, & t \geq 3. \end{cases}$$

(c) Use the method of Laplace transforms and your answers to (a) and (b) to find a solution to the initial value problem

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = h(t), \quad y(0) = 0, \ \left.\frac{dy}{dt}\right|_{t=0} = 1,$$

where $h(t)$ is as defined in (a).

You may use without proof the result

$$\frac{1}{s^2(s+1)(s+2)} = -\frac{3}{4s} + \frac{1}{2s^2} - \frac{1}{4(s+2)} + \frac{1}{s+1}.$$

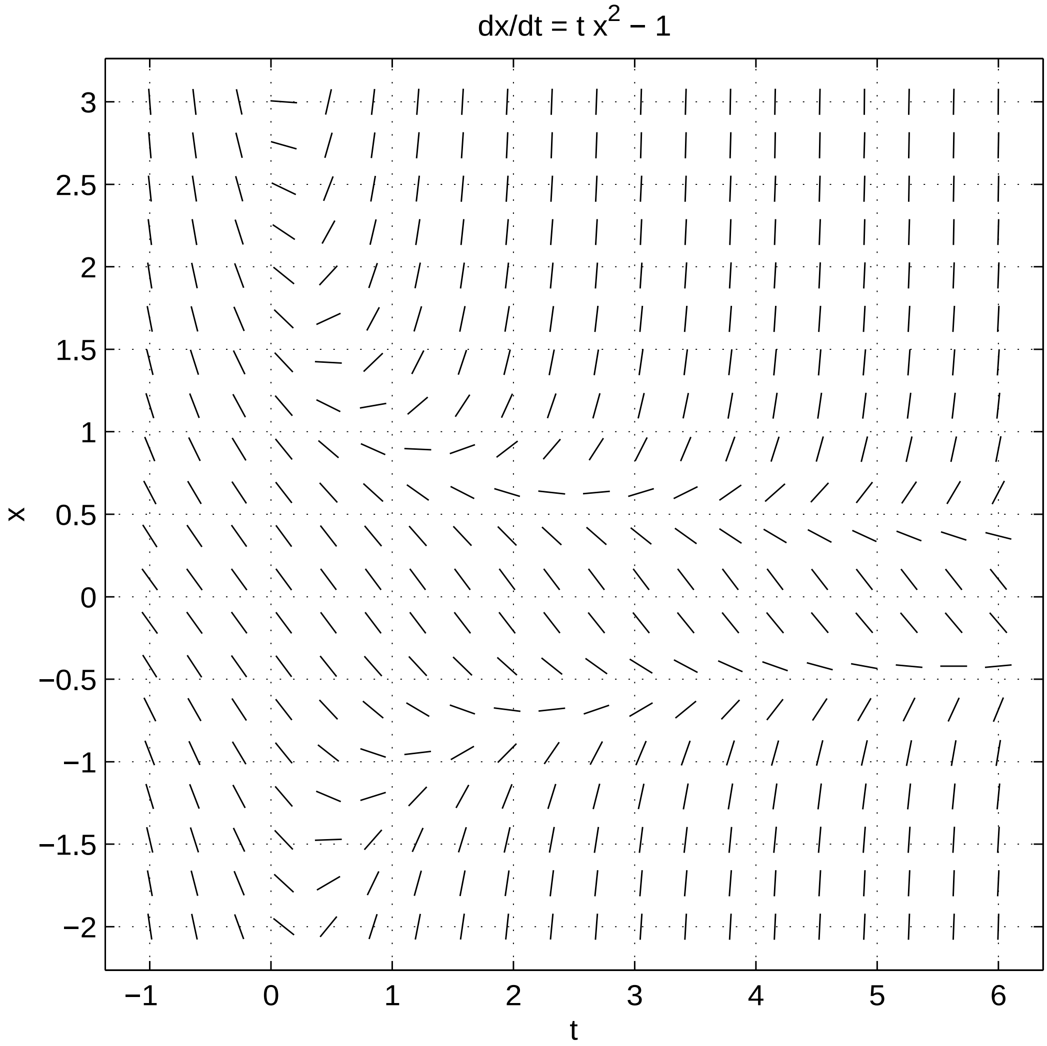
A brief table of Laplace transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
$e^{at}t^n,\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{U}_a,\ a \geq 0$	$\frac{e^{-as}}{s}$
$f(t-a)\mathcal{U}_a,\ a \geq 0$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\frac{d^n f}{dt^n}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$

Candidate's Name: ______________________ ID No: ____________

TIE THIS ANSWER SHEET TO YOUR SCRIPT BOOK

Answer sheet for Question 3



Answer sheet for Question 7