Maths 260 Lecture 13

- **Topic for today:**
  Linear equations again
- **Reading for this lecture:** BDH Section 1.9
- **Suggested exercises:** BDH Section 1.9, #1,3,9,13
- **Reading for next lecture:** BDH Section 2.1
- **Today's handouts:** Tutorial 5 question sheet

Linear Differential Equations

Recall that a first order DE is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

In the last lecture, we found the general solution to such DEs by combining the general solution to the associated homogeneous DE with a particular solution to the nonhomogeneous DE.

We had to guess the particular solution, but there is sometimes a better way.

Another method for finding solutions to linear DEs

First rewrite the DE as

$$\frac{dy}{dt} + g(t)y = b(t)$$

where $g(t) = -a(t)$.

Multiply both sides of the DE by $\mu(t)$, which is an unknown, nonzero function that will be determined later.

The DE now takes the form

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

Now assume we can pick $\mu(t)$ so that

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \frac{d}{dt}(\mu(t)y)$$

Then

$$\frac{d}{dt}(\mu(t)y) = \mu(t)b(t)$$

Integrating both sides with respect to t gives

$$\mu(t)y(t) = \int \mu(t)b(t)dt$$
$$\Rightarrow y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)dt$$

If we can find such a function $\mu(t)$ and do the integration then we can find $y(t)$.

The function $\mu(t)$ is called an **integrating factor.**

Finding the integrating factor, $\mu(t)$

We want $\mu(t)$ such that

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

After cancelling terms, this is:

$$\mu(t)g(t)y = \frac{d\mu}{dt}y \quad \Rightarrow \quad \frac{d\mu}{dt} = \mu(t)g(t)$$

This is a separable DE for $\mu(t)$, and we can solve it:

$$\int \frac{d\mu}{\mu} = \int g(t)dt$$

$$\Rightarrow \quad \ln|\mu| = \int g(t)dt$$
$$\Rightarrow \quad \mu(t) = \pm \exp\left(\int g(t)dt\right)$$

Different choices of the constant of integration will give slightly different integrating factors $\mu(t)$ but all choices give a valid integrating factor.

We usually pick the easiest, i.e., pick the positive sign and a zero constant of integration.

Summary of method: To find a solution to

$$\frac{dy}{dt} + g(t)y = b(t)$$

find the integrating factor

$$\mu(t) = \exp\left(\int g(t)dt\right)$$

Then the solution to the DE is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)dt$$

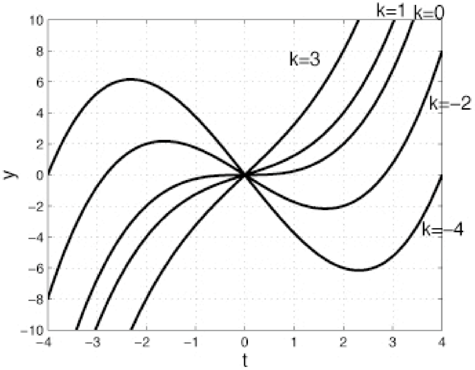
Example 1: Find a one-parameter family of solutions to

$$\frac{dy}{dt} = \frac{y}{t} + t^4, \qquad t > 0$$

We find the DE has a one-parameter family of solutions

$$y(t) = \frac{t^5}{4} + kt$$

where k is an arbitrary constant. These solutions are plotted below for various k.



Example 2: Find a solution to the IVP

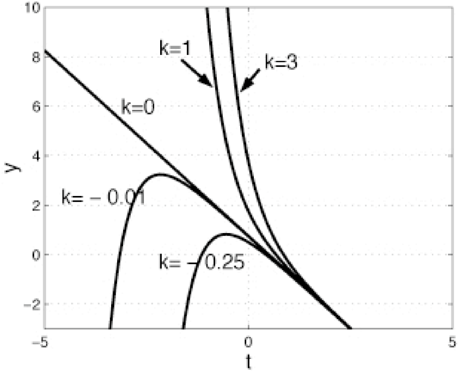
$$\frac{dy}{dt} = -2y - 3t, \qquad y(0) = \frac{1}{2}.$$

We find the DE has a one-parameter family of solutions

$$y(t) = -\frac{3}{2}t + \frac{3}{4} + ke^{-2t}.$$

The choice $y(0) = 1/2$ determines $k = -1/4$.

Some solutions to the DE including the solution to the IVP are plotted below.



Example 3: Find a solution to the IVP

$$\frac{dy}{dt} = 1 + 2yt, \qquad y(0) = 1$$

Important ideas from today:

- A first order DE is linear if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

- Explicit solutions to linear DEs can sometimes be found. First compute the integrating factor (if possible):

$$\mu(t) = \exp\left(\int g(t)dt\right)$$

where $g(t) = -a(t)$.

Then the solution to the DE is

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)b(t)dt$$