Maths 260 Lecture 24

- **Topic for today:** Bifurcations in linear systems
- **Reading for this lecture:** BDH Section 3.7
- **Suggested exercises:** BDH Section 3.7; 3, 5, 7, 9
- **Reading for next lecture:** BDH Section 5.1
- **Today's handouts:** Tutorial 9

Putting it all together - bifurcations in linear systems

- What are the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$?
- Characteristic equation is

$$\begin{aligned} \lambda^2 - (a+d)\lambda + ab - cd &= 0 \\ \lambda^2 - T\lambda + D &= 0 \end{aligned}$$

- If eigenvalues are λ_1 and λ_2 can also write characteristic equation as

$$\begin{aligned} (\lambda - \lambda_1)(\lambda - \lambda_2) &= 0 \\ \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 &= 0 \end{aligned}$$

which leads to the following result.

Result on eigenvalues

For any matrix $\mathbf{A}$,

- $\det(\mathbf{A})$ = product of the eigenvalues of $\mathbf{A}$
- $\text{trace}(\mathbf{A})$ = sum of the eigenvalues of $\mathbf{A}$

Example 1: Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & -5 \\ 1 & 3 \end{pmatrix}$$

- If $\mathbf{A}$ is a 2 by 2 matrix, the signs of $\det(\mathbf{A})$ and $\text{trace}(\mathbf{A})$ tell us a lot about the type of the equilibrium at the origin for the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

- For example, if $\mathbf{A}$ is a 2 by 2 matrix with $\det(\mathbf{A}) < 0$, then there is one positive, and one negative eigenvalue, so the origin is a saddle.

- If $\det(\mathbf{A}) > 0$, then the eigenvalues have real parts of the same sign.

- If $\text{trace}(\mathbf{A}) > 0$ both eigenvalues have positive real part, and if $\text{trace}(\mathbf{A}) < 0$ both eigenvalues have negative real part.

Example 2:

- Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 1 & 2 \\ a & 0 \end{pmatrix} \mathbf{Y}$$

  where a is a parameter.
- Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

- We find that the eigenvalues of matrix $\mathbf{A}$ are

$$\lambda_1 = \frac{1 + \sqrt{1 + 8a}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{1 + 8a}}{2}$$

  and that $\det(\mathbf{A}) = -2a$, $\text{trace}(\mathbf{A}) = 1$.
- The following qualitatively distinct cases can occur, depending on a.
  - **Case 1:** If $1 + 8a < 0$, the eigenvalues of $\mathbf{A}$ are complex.

- **Case 2:** If $1 + 8a > 0$, the eigenvalues of $\mathbf{A}$ are real.
    - **Case 2a:** If $a > 0$, then $\det(\mathbf{A}) < 0$, so there is one positive eigenvalue and one negative eigenvalue, and so the origin is a saddle.

- **Case 2b:** If $-1/8 < a < 0$, then $\det(\mathbf{A}) > 0$, so the eigenvalues are of the same sign as each other (and real). Since $\text{trace}(\mathbf{A}) > 0$ the eigenvalues are both positive. The origin is a source.

A source or sink arising from real eigenvalues is often called a node.

Transitional values of a

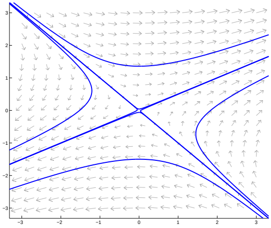
- If $a = -\frac{1}{8}$,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{8} & 0 \end{pmatrix}$$

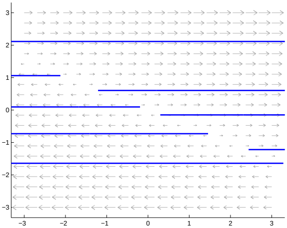
- In this case, the eigenvalues of $\mathbf{A}$ are $1/2$ (twice) with just one linearly independent eigenvector $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

- If $a = 0$,
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$
- In this case, the eigenvalues of $\mathbf{A}$ are 0 and 1, with eigenvectors
$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
respectively.

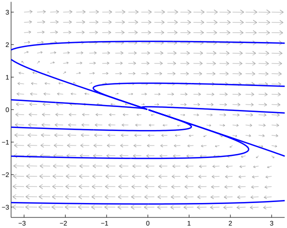
Representative phase portraits plotted with pplane:



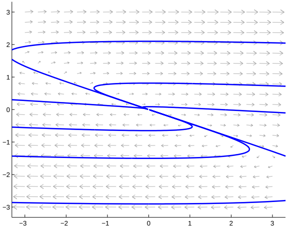
$a = 1$



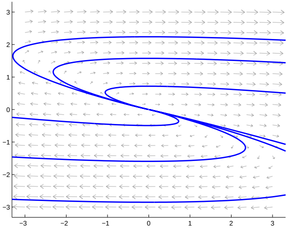
$a = 0$



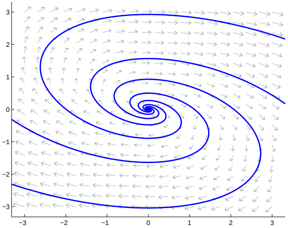
$a = -1/16$



$a = -1/16$ (again)



$a = -1/8$



$a = -1$

Example 3:

- Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix} \mathbf{Y}$$

  where a is a parameter.
- Determine the type of equilibrium at the origin for all values of a. Sketch the phase portrait for representative values of a.

- $a < -2$, e.g. $a = -3$

- $-2 < a < 0$, e.g. $a = -1$

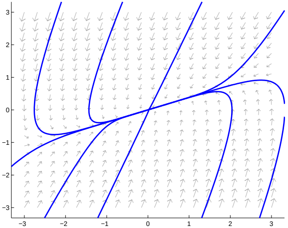
- $0 < a < 2$, e.g. $a = 1$

- $2 < a$, e.g. $a = 3$

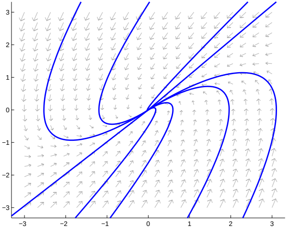
Transitional values of a:

- $a = -2$
- $a = 0$
- $a = 2$

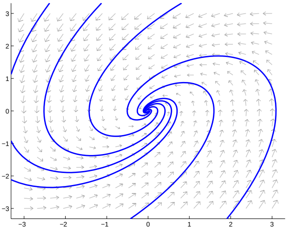
Representative phase portraits plotted with pplane:



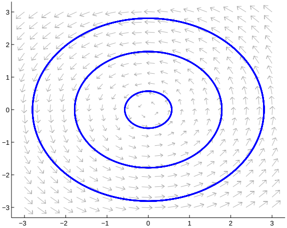
$a = -3$



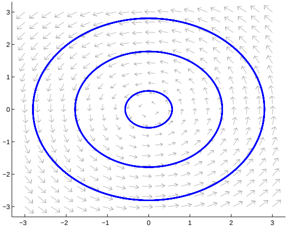
$a = -2$



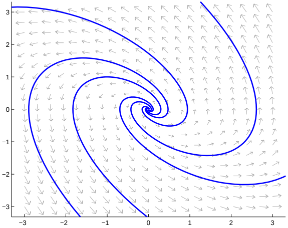
$a = -1$



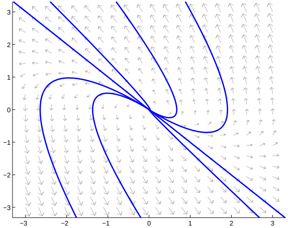
$a = 0$


3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

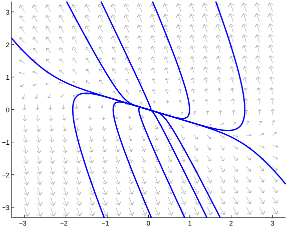
$a = 0$


3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

$a = 1$


3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

$a = 2$


3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

$a = 3$

Important ideas from today

From these two examples we see how the transitional cases arise as a parameter is varied:

- a centre occurs as a spiral sink changes to a spiral source, or vice versa;
- an improper node (i.e. two equal eigenvalues with only one linearly independent eigenvector) occurs when a spiral sink (or source) turns into a real sink (or source), or vice versa;
- a linear system with a zero eigenvalue occurs when a saddle turns into a sink or source, or vice versa.