Maths 260 Lecture 11

- **Topics for today:**
  More on bifurcations
- **Reading for this lecture:** BDH Section 1.7
- **Suggested exercises:** BDH Section 1.7, #11
- **Reading for next lecture:** BDH Section 1.8
- **Today's handouts:** None

We are interested in one-parameter families of autonomous DEs, which can be written in the form

$$\frac{dy}{dt} = f_k(y).$$

We look for bifurcations, i.e., changes in the qualitative behaviour of solutions as the parameter k is varied.

General result about bifurcations

Bifurcations usually do not happen, i.e., a small change in the parameter usually leads to only a small change in the behaviour of solutions, with no overall qualitative change in behaviour.

To be precise, if

$$\frac{dy}{dt} = f_k(y),$$

and if

$$\frac{\partial f}{\partial k} \text{ and } \frac{\partial f}{\partial y}$$

exist and are continuous for all values of k and y, then a small change in k gives a small change in the graph of $f_k(y)$.

Example 1: Suppose the DE

$$\frac{dy}{dt} = f_k(y)$$

has a source at $y = y_0$ when $k = k_0$, with

$$\left.\frac{df_{k_0}}{dy}\right|_{y=y_0} > 0.$$

What is the effect on the qualitative behaviour of solutions of changing k by a small amount?

A bifurcation where the number or type of equilibrium solutions changes can only occur at $k = k_0$ if

$$f_{k_0}(y_0) = 0 \qquad \text{and} \qquad \frac{df_{k_0}}{dy}(y_0) = 0,$$

i.e., when the linearization theorem does not work.

Example 2: Draw the bifurcation diagram for the family of equations

$$\frac{dy}{dt} = ky - y^3$$

Example 3: Draw the bifurcation diagram for the family of equations

$$\frac{dy}{dt} = ky + y^2$$

Important ideas from today:

- Bifurcations are special: a small change in parameter does not usually result in a qualitative change in the behaviour of solutions.
- A bifurcation at which the number or type of equilibrium solutions changes can only occur at $k = k_0$ if

$$f_{k_0}(y_0) = 0 \qquad \text{and} \qquad \frac{df_{k_0}}{dy}(y_0) = 0,$$

  i.e., at a parameter value where the linearization theorem does not work.