THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2008
Campus: City

MATHEMATICS

Differential Equations

(Time allowed: TWO hours)

NOTE:
- There are SEVEN questions. Answer ALL questions.
- Total number of marks is 100.
- Show all your working.

1. (16 marks)

(a) Find the solution of the initial value problem

$$\frac{dy}{dt} = -2y + e^t, \quad y(0) = 1.$$

(b) Find the general solution to differential equation

$$\frac{dy}{dt} = y \cos t.$$

(c) Find the general solution to differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = t.$$

2. (15 marks)
Consider the initial value problem

$$\frac{dy}{dt} = y^2 - 2t, \quad y(0) = 0.5.$$

A direction field is provided on the yellow answer sheet attached to the back of the question paper. Use the direction field for your answer to part(a) of this question. Attach the yellow answer sheet to your answer book.

(a) On the direction field, sketch carefully the solution of the initial value problem. Estimate the value of $y(1)$.

(b) Perform one step of Euler's method to calculate an approximation to $y(1)$.

(c) Perform two steps of Euler's method to calculate an approximation to $y(1)$.

(d) Using your answer in (a), estimate the error in each of the approximations in (b) and (c).

(e) Comment on the errors you found in (d). Are they what you expect?

(f) Use the Existence and Uniqueness Theorems to show that the initial value problem has a unique solution.

3. (18 marks)
Consider the one-parameter family of differential equations

$$\frac{dy}{dt} = y(y + \alpha - 1).$$

(a) For the case $\alpha = 2$:
  (i) Find all equilibrium solutions and determine their type (e.g., sink, source).
  (ii) Sketch the phase line.

(b) Repeat (a) for the case $\alpha = 1$.

(c) Repeat (a) for the case $\alpha = -1$.

(d) Now let α vary.
  (i) Locate the equilibrium solutions and determine their type as a function of α.
  (ii) Sketch the bifurcation diagram and find the bifurcation point. Be sure to label the main features of the bifurcation diagram. Show all your working.

4. (6 marks)
Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -4 \\ a & 0 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a saddle.

(b) Find all values of a for which the equilibrium solution at $(x, y) = (0, 0)$ is a spiral source.

5. (8 marks)
Consider the following system of differential equations:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find the general solution to the differential equation.

(b) Sketch the corresponding phase portrait.

6. (25 marks) Consider the following system of equations:

$$\frac{dx}{dt} = -x + y - y^2,$$
$$\frac{dy}{dt} = -2x.$$

A grid is provided on the yellow answer sheet attached to the back of the question paper. Use the grid for your answer to parts (b) and (c) of this question. Attach the yellow answer sheet to your answer book.

(a) Find all equilibrium solutions and determine their type (e.g., spiral source, saddle). For each equilibrium you find, draw a phase portrait showing the behaviour of solutions near that equilibrium.

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system. Your phase portrait should show the behaviour of solutions near the equilibria, and should show various solution curves *including* those passing through the following initial conditions:

(i) $(x(0), y(0)) = (0, 0.5)$;

(ii) $(x(0), y(0)) = (0, 2)$;

(iii) $(x(0), y(0)) = (-1, 1)$.

Make sure you show clearly where solution curves go as $t \to \infty$.

7. (12 marks) A large oil tanker floats on the sea. Suppose the tanker is given a small downward displacement z. The upward force on the tanker is equal to the weight of water displaced (Archimedes Principle) and a simple model for the motion of the tanker is

$$\frac{d^2z}{dt^2} + \omega^2 z = 0,$$

where ω is a real constant.

(a) Find the general solution for z. Express your answer in terms of real-valued functions.

(b) Suppose now that a small mouse dances on the deck of the tanker producing a downwards force $m \sin \omega t$. Discuss the long term behaviour of solutions, and what this means for the motion of the tanker. You do not have to find any solutions to answer this part of the question.

(c) A more realistic model for the motion of the tanker (without the mouse) might be

$$\frac{d^2z}{dt^2} + k\frac{dz}{dt} + \omega^2 z + az^3 = 0,$$

where k and a are constants.

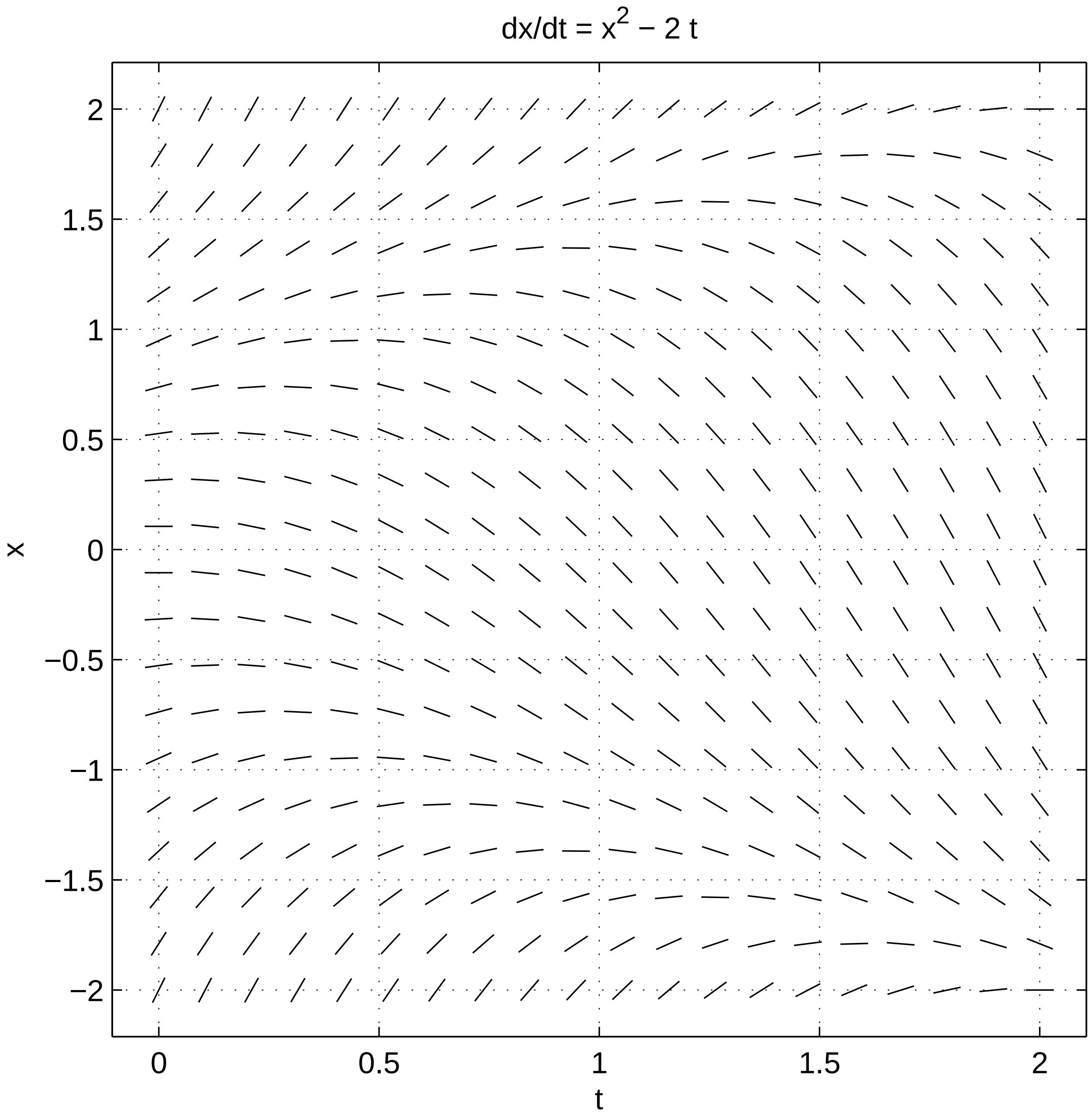
(i) Briefly describe what the new terms might represent physically.

(ii) Briefly describe the methods you could use to get information about solutions to this model. You do not need to do any calculations to answer this part of the question.

Candidate's Name: ________________________________ ID No: ______________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

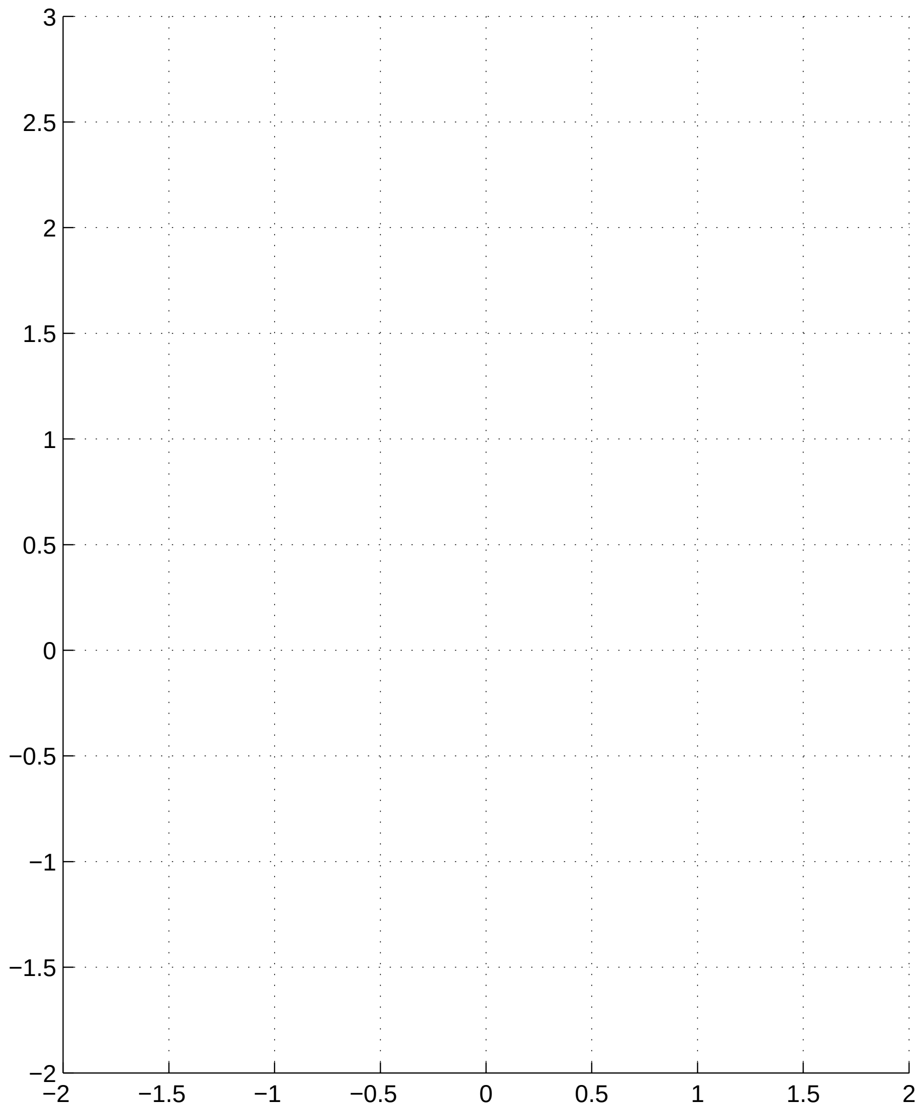
Answer sheet for Question 2



Candidate's Name: ____________________ ID No: __________

TIE THIS ANSWER SHEET TO YOUR ANSWER BOOK

Answer sheet for Question 6



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