Maths 260 Lecture 9

- **Topics for today:**
  Classification of equilibria
  Linearization
- **Reading for this lecture:** BDH Section 1.6, pp 86-91
- **Suggested exercises:** BDH Section 1.6, #1,3,5,7,13,15,17
- **Reading for next lecture:** BDH Section 1.7
- **Today's handouts:** None

Classifying Equilibria

To draw the phase line for an autonomous DE

$$\frac{dy}{dt} = f(y)$$

we need to know the positions of all equilibria, the intervals of y where $f(y) > 0$, and the intervals of y where $f(y) < 0$.

Example 1: Draw the phase line for the DE

$$\frac{dy}{dt} = (1 - y^2)y$$

If f is continuous, the sign of f can only change at y-values where $f(y) = 0$, i.e., at equilibria.

Thus, the positions of the equilibria and the behaviour of solutions near each equilibrium is all we need to know to draw the phase line.

Example 2: Draw the phase line for a DE

$$\frac{dy}{dt} = f(y)$$

which has solutions with the properties:

- there are two equilibria, at $y = 0$ and $y = 2$;
- solutions started near $y = 0$ tend to $y = 0$ as $t \to \infty$;
- solutions started near $y = 2$ always decrease.

We classify equilibria according to the behaviour of nearby solutions.

An equilibrium $y(t) = a$ is a **sink** if any solution with initial condition sufficiently close to $y = a$ moves toward $y = a$ as t increases.

An equilibrium $y(t) = b$ is a **source** if any other solution with initial condition sufficiently close to $y = b$ moves away from $y = b$ as t increases (which means nearby solutions diverge from $y = b$ as t increases.)

An equilibrium that is neither a sink nor a source is called a **node.**

Example 3: For the DE

$$\frac{dy}{dt} = y(3 + y)$$

find all equilibrium solutions and determine their types.
Draw the phase line.

Example 4: For the DE

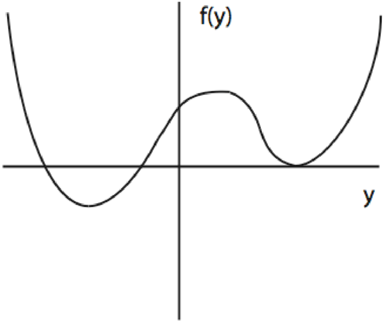
$$\frac{dy}{dt} = y(y+2)^2$$

find all equilibrium solutions and determine their types.
Draw the phase line.

Example 5: For the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ has the graph shown below, find all equilibrium solutions and determine their types. Draw the phase line.



Linearization

If y_0 is an equilibrium solution of

$$\frac{dy}{dt} = f(y)$$

and is a sink, then

- $f(y) > 0$ if $y < y_0$
- $f(y) < 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a decreasing function near y_0.

If y_0 is an equilibrium solution of

$$\frac{dy}{dt} = f(y)$$

and is a source, then

- $f(y) < 0$ if $y < y_0$
- $f(y) > 0$ if $y > y_0$
- $f(y_0) = 0$

So $f(y)$ is a increasing function near y_0.

Linearization Theorem

Suppose that $y = y_0$ is an equilibrium point of the DE

$$\frac{dy}{dt} = f(y)$$

where $f(y)$ and df/dy are both continuous functions of y.

- If $f'(y_0) < 0$, then y_0 is a sink.
- If $f'(y_0) > 0$, then y_0 is a source.
- If $f'(y_0) = 0$ or if $f'(y_0)$ does not exist, then we need additional information to determine the type of y_0. In this case the equilibrium may be a sink, a source, or a node.

Here, $f'(y_0)$ means df/dy evaluated at $y = y_0$.

Example 6: For the DE

$$\frac{dy}{dt} = y^2(y-2)(y+2)$$

find all equilibrium solutions and classify them using the linearization theorem.

Note that $y = y_0$ is not necessarily a node in the case that

$$\left.\frac{df}{dy}\right|_{y=y_0} = 0$$

Example 7: Determine the type of equilibrium at $y = 0$ for the DE

$$\frac{dy}{dt} = y^3.$$

Example 8: Consider the following population model:

$$\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{200}\right)\left(\frac{P}{50} - 1\right)$$

Classify the equilibria, draw the phase line and sketch some solutions for P.

Important ideas from today:

- An equilibrium solution is classified as a sink, source, or node depending on the behaviour of nearby solutions.
- Linearization - we can sometimes use the quantity df/dy evaluated at an equilibrium solution to determine the type of that equilibrium.
- We have learnt two methods for drawing phase lines:
  - by using a table to work out the intervals of y for which dy/dt is positive and intervals where it is negative, then sketching the phase line from this information;
  - using linearisation to determine the type of each equilibrium and then sketching the phase line.

It is usually quicker to use linearisation, and to use the first method only if linearisation fails.