MATHS 270FC: Numerical Computation

Assignment 2

Questions and sample answers

Due	4:00pm, Friday April 15 at the Student Resource Centre.
Marks	20 is full marks.

Mechanical

1. Apply three iterations of

   a) [2 marks] Bisection

   b) [2 marks] Secant

   c) [2 marks] Newton Raphson

to $3x^2 - \exp(x) = 0$. For a) start with the bracket $[0, 1]$; for b) use $x^{(0)} = 0$, $x^{(1)} = 1$, and for c) use $x^{(0)} = 1$.

Ans: For the method of Bisection, we have

k	a	b	f_a	f_b	c	f_c
2	0	1	-1	0.2817	0.5	-0.8987
3	0.5	1	-0.8987	0.2817	0.75	-0.4295
4	0.75	1	-0.4295	0.2817	0.875	-0.1020

For Secant, we have

$$x^{(k)} = x^{(k-1)} - \frac{x^{(k-1)} - x^{(k-2)}}{f_{k-1} - f_{k-2}} f_{k-1}$$

Since $f(x^{(0)}) = f(0) = -1$ and $f(x^{(1)}) = f(1) = 0.2817$, the Secant method gives

$$\begin{aligned}
x^{(2)} &= 1 - \frac{1 - 0}{0.2817 - (-1)} 0.2817 = 0.7802 \\
x^{(3)} &= 0.7802 - \frac{0.7802 - 1}{-0.3557 - 0.2817} (-0.3557) = 0.9029 \\
x^{(4)} &= 0.9029 - \frac{0.9029 - 0.7802}{-0.0212 - (-0.3557)} (-0.0212) = 0.9107
\end{aligned}$$

For Newton Raphson, we have

$$x^{(k)} = x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})}$$

Since $f(x) = 3x^2 - \exp(x)$, we have $f'(x) = 6x - \exp(x)$. Then, starting with $x^{(0)} = 1$, we get $x^{(1)} = 0.914155$, $x^{(2)} = 0.910018$, $x^{(3)} = 0.910008$.

2. [4 marks] Solve

$$\begin{bmatrix} 1 & 2 & 2 \\ -3 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} x = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

using LU factoring with maximal column pivoting. You may use your calculator but not matlab. Show your intermediate results and work to at least four decimal places.

Ans: The coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 2 \\ -3 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

The first step is to set $n_p = [1, 2, 3]^T$. Next, we search down the first column for the element of largest magnitude. This is the second one, so we interchange the first and second rows. This gives the matrix

$$\begin{bmatrix} -3 & 2 & 4 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

We also interchange the first and second elements of n_p to give $[2, 1, 3]^T$. Now do the elementary row operations $r_2 = r_2 - (-1/3)r$ (no operation is needed for the third row - why?). This gives the matrix

$$\begin{bmatrix} -3.0000 & 2.0000 & 4.0000 \\ -0.3333 & 2.6667 & 3.3333 \\ 0.0000 & 1.0000 & 2.0000 \end{bmatrix}$$

where I have stored the multiplier m_{21} in the matrix (m_{31} is zero). This completes the first column.

For the second column we do not have to do any row interchanges (why?). So we just do the elementary row operation $r_3 = r_3 - (2.0000/2.6667)r_2$. This gives the matrix

$$\begin{bmatrix} -3.0000 & 2.0000 & 4.0000 \\ -0.3333 & 2.6667 & 3.3333 \\ 0.0000 & 0.3750 & 0.7500 \end{bmatrix}$$

This finishes the factoring. The next step is the forward substitution i.e. solve $Ly = Pb$. By inspection, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.3333 & 1 & 0 \\ 0.0000 & 0.3750 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} -3.0000 & 2.0000 & 4.0000 \\ 0 & 2.6667 & 3.3333 \\ 0 & 0 & 0.7500 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The system $Ly = Pb$ is then easily solved ($b = [-3, 1, 2]^T$, this was given in the question). Get

$$y = [1.0000, -2.6667, 3.0000]^T$$

The last step is to solve $Ux = y$ by backsubstitution. Get

$$x = [1.0000, -6.0000, 4.0000]^T$$

A bit more thinking

3. [6 marks] Find all the real roots of $x^4 - 11.6x^3 + 46.86x^2 - 76.676x + 41.6185$ using any combination of Bisection, Secant and Newton Raphson. Your values of x for the roots must be within 10^{-8} of the true answer. You can use matlab to help you with the calculations. Include enough information in your answer to ensure it can be marked.

Ans: Let $f(x) = x^4 - 11.6x^3 + 46.86x^2 - 76.676x + 41.6185$. The first thing I did was write the following matlab function

```
function y = f(x)

   y = x.^4 - 11.6*x.^3 + 46.86*x.^2 - 76.676*x + 41.6185;
```

I then used this function to plot $f(x)$. It took me several plots to find that $f(x)$ had four real roots and that they were near 1, 2, 3.5 and 4.5.

I decided to use Newton Raphson. This meant I needed a matlab function for $f'(x)$:

```
function y = f(x)

   y = 4*x.^3 - 3*11.6*x.^2 + 2*46.86*x - 76.676;
```

When I started Newton Raphson from $x^{(0)} = 1$, I found the iterations converged to 1.09999999999999 after four iterations. I concluded one root was 1.1 to eight decimal places.

When I started Newton Raphson from $x^{(0)} = 2$, I found the iterations converged to 2.300000000000000 after four iterations. I concluded another root was 2.3 to eight decimal places.

When I started Newton Raphson from $x^{(0)} = 3.5$, I found $f(3.5) = 1.42 \times 10^{-14}$. I concluded another root was 3.5 to eight decimal places.

When I started Newton Raphson from $x^{(0)} = 4.5$, I found the iterations converged to 4.70000000000003 after five iterations. I concluded the final root was 4.7 to eight decimal places.

4. [4 marks] Use Newton's method to find an approximation for λ, accurate to within 10^{-4}, for the population equation

$$1,564,000 = 1,000,000e^{\lambda} + \frac{435,000}{\lambda}(e^{\lambda} - 1).$$

Include enough information in your answer to ensure it can be marked.

Ans: I decided to use matlab. I first typed in the functions for $f(x)$ and $f'(x)$, where $f(x) = 1564000 - 1,000,000e^{x} + \frac{435,000}{x}(e^{x} - 1)$.

```
function y = f(x)

y = 1564000 - 1000000*exp(x) - 435000./x.*(exp(x)-1);

function y = fp(x)

y = -1000000*exp(x) + 435000./x.^2.*(exp(x)-1) - 435000./x.*exp(x);
```

Next I plotted $f(x)$ and found the root was near $x = 0.1$. I then did the following in matlab (you could have used a loop or the `nr` function I gave in class)

```
x = 0.1

x =

    0.1000

x = x - f(x)/fp(x)
```

```
x =

    0.1010

format long e
x

x =

     1.009983993922773e-01

x = x - f(x)/fp(x)

x =

     1.009979296858536e-01
```

The last two iterates agree to four decimal places and the required value of λ is 0.1010 to four decimal places.

Possible exam question

Let $f(x) = x - \cos x$.

a) [2 marks] Apply two iterations of the Bisection method to $f(x) = 0$. Start with $x^{(0)} = 0$ and $x^{(1)} = 1$.

Ans:

k	a	b	f_a	f_b	c	f_c
2	0	1	-1	0.4597	0.5	-0.3776
3	0.5	1	-0.3776	0.4597	0.75	0.183

b) [2 marks] Apply two iterations of the Newton-Raphson method to $f(x) = 0$. Start with $x^{(0)} = 0$.

Ans: The Newton Raphson formula is

$$\begin{aligned} x^{(k)} &= x^{(k-1)} - \frac{f(x^{(k-1)})}{f'(x^{(k-1)})} \\ &= x^{(k-1)} - \frac{x^{(k-1)} - \cos(x^{(k-1)})}{1 + \sin(x^{(k-1)})} \end{aligned}$$

When I applied the above formula starting from $x^{(0)} = 0$, I got $x^{(1)} = 1$ and $x^{(2)} = 0.7504$.

c) [2 marks] Suppose you have to find the root of $f(x) = 0$ to six decimal places. Would you use the Newton-Raphson or Bisection method? Justify your answer.

Ans: I would expect two answers. One answer is:

I would use the method of Bisection, because there is no guarantee Newton Raphson will converge.

The other answer is:

I would use Newton Raphson because the calculations in part a) suggest the method is converging and it will require fewer calculations to find the solution to six decimal places.

Notes for this question

a) In the final exam, you would have 15-20 minutes to do the above question.

b) The marks given for each part of the question are the marks for this assignment. Each part would be worth more marks in the final exam.

Assessment for the course

1. Assignments: 15% of the final mark, four assignments each worth 3 3/4%, each has bonus marks.
2. Mid-semester test: 15% of the final mark, no bonus marks. There will be no questions on matlab.
3. Final exam: 70% of the final mark, no bonus marks, each section of the course will be examined, there will be one question on matlab.
4. General: To get good marks it is important to show your (correct) understanding of the appropriate course material. There is no plussage.

Challenge question

[4 marks] Solve the 50 problems $f_i(\theta) = 0$, $i = 1, \ldots, 50$, where

$$f_i(\theta) = \theta - \sin(\theta) - \frac{i}{100}$$

using a total of no more than 300 evaluations of $f(\theta)$ or its derivatives. You must solve the problems in ascending order of i and you must use the initial estimate $\theta = 0.01$ for the first problem.

Ans: If you wish to see a sample solution, come and see me.