1. (a) (**3 marks**) "n is odd but $n \neq 5q+1$ and $n \neq 5q+3$ for all integers q".

(b) (**2 marks**) "If $n \neq 5q+1$ and $n \neq 5q+3$ for all integers q, then n is even".

(c) (**2 marks**) "If $n = 5q+1$ or $n = 5q+3$ for some integer q, then n is odd".

(d) (**3 marks**) The converse of $A(n)$ is NOT TRUE.
Take $n = 6$. Then $n = 5+1$, but n is not odd. So $n = 6$ is a conterexample to the converse of $A(n)$.

2. (a) (**7 marks**)

$$\begin{aligned} f(a) = f(a') &\iff \frac{4a}{a-5} = \frac{4a'}{a'-5} \\ &\iff \frac{4(a-5)+20}{a-5} = \frac{4(a'-5)+20}{a'-5} \\ &\iff 4 + \frac{20}{a-5} = 4 + \frac{20}{a'-5} \\ &\iff \frac{20}{a-5} = \frac{20}{a'-5} \\ &\iff a'-5 = a-5 \\ &\iff a' = a, \end{aligned}$$

so f is one-one.
Now we show that f is onto.
For any $b \in B$, let $a = \frac{5b}{b-4}$. Since $b \neq 4$, it follows that $a \in \mathbb{R}$. If $a \notin A$, then $a = 5$, i.e. $b = b-4$ or $0 = -4$, which is impossible. Thus $a \in A$ and

$$f(a) = \frac{4\frac{5b}{b-4}}{\frac{5b}{b-4} - 5} = \frac{20b}{5b - (5b-20)} = b,$$

so f is onto.

(b) (**3 marks**) As shown above $f^{-1} : B \to A$ is given by $f^{-1}(x) = \frac{5x}{x-4}$

3. (a) (**6 marks**) $\sim$ is **reflexive** because $4 \mid (x+3x) = 4x$ for all $x \in S$.
$\sim$ is **symmetric**. Suppose $x \sim y$. Then $4 \mid (x+3y) \iff x+3y = 4t$ for some $t \in \mathbb{Z}$, and so

$$y + 4x = 4y + 4x - (x+3y) = 4(y+x-t).$$

Thus $4 \mid (y+3x)$ and $y \sim x$.
$\sim$ is **transitive**. Suppose $(x \sim y) \wedge (y \sim z)$ for some $x, y, z \in S$. Then $4 \mid (x+3y)$ and $4 \mid (y+3z)$, so that

$$4 \mid (x+3z) = (x+3y) + (y+3z) - 4y.$$

Thus $x \sim z$.

(b) (**4 marks**) $[0] = \{x \in S : 4 \mid x + 3 * 0 = x\} = \{x \in S : 4 \mid x\} = \{-4, 0, 4\}$, $[1] = \{x \in S : 4 \mid x + 3\} = \{-7, 1, 5\}$, $[2] = \{x \in S : 4 \mid x + 6\} = \{x \in S : 4 \mid x + 2\} = \{-6, -2, 2\}$ and $[3] = \{x \in S : 4 \mid x + 9\} = \{x \in S : 4 \mid x + 1\} = \{7\}$.

4. (a) (**7 marks**) **Proof:** Let $n \in \mathbb{N}$ and let $P(n)$ be the statement

$$\text{“}1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}\text{''}.$$

Base Case: $P(1)$ is true, since $LHS = 1 = 2 - \frac{1}{1} = RHS$.
Inductive Step: Suppose $k \geq 1$ and $P(k)$ is true. So

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.$$

Consider $P(k+1)$.

$$\begin{aligned}
LHS &= (1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}) + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \qquad \text{by the inductive hypothesis} \\
&= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\
&= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\
&= 2 - \frac{k^2 + k + 1}{(k^2 + k)(k+1)} \\
&= 2 - (1 + \frac{1}{k^2 + k})\frac{1}{k+1} \\
&\leq 2 - \frac{1}{k+1} \\
&= RHS,
\end{aligned}$$

that is, $P(k+1)$ is true. By induction, $P(n)$ is true for all $n \in \mathbb{N}$.

(b) (**3 marks**) Suppose, for a contradiction that there is a partial order Q on A such that $R \subseteq Q$. Since $(1,2) \in R$ and $(2,1) \in R$, it follows that $(1,2) \in Q$ and $(2,1) \in Q$. But Q is antisymmetric, so $1 = 2$, which is impossible.