1. (a) (**3 marks**) Since $x * y = y \in X$ for any $x, y \in X$, it follows that $*$ is a binary operation of X. The following is its Cayley table.

$*$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	a	b	c	d
d	a	b	c	d

Since 18 is not divisible by 22, it follows that $946x + 374y = 18$ has no integer solution.

(b) (**3 marks**) For all $x, y, z \in X$,

$$x * (y * z) = y * z = z = (x * y) * z,$$

so that $*$ is associative. Since $a * b = b \neq a = b * a$, it follows that $*$ is non-commutative.

(c) (**2 marks**) Suppose $(X, *)$ has the identity $e \in X$. For any $x \in X$ and $y \in X$ with $y \neq x$. Then $x * y = y \neq x$, so $e \neq x$, a contradiction.

(d) (**2 marks**) Since $(X, *)$ has no identity, so it is not a group.

2. (a) (**3 marks**) Since $hd = \varphi$ and $dh = \phi \neq \varphi$, it follows that $dh \neq hd$ and D_4 is non-Abelian.

(b) (**4 marks**) If $x \in D_4$ such that $x^{-1} = x$, then $x^2 = x \cdot x^{-1} = e$. Thus

$$x \in \{e, \pi, h, v, d, d'\}.$$

(c) (**4 marks**) Let $H = \{e, \pi\}$ and $K = \{e, h\}$. Then

$*$	e	π
e	e	π
π	π	e

Thus $x * y \in H$ for any $x, y \in H$. Since $x^{-1} = x \in H$, it follows by Two-step test that H is a subgroup of G. Similarly, $K \leq G$ and H and K are two distinct subgroups of order 2.

OR use finite subgroup test $H \leq G \iff (H \neq \emptyset) \wedge (\forall\, x, y \in H)(x * y \in H)$.

(d) (**5 marks**) Let $L = \{e, \varphi, \pi, \phi\}$. By the Cayley Table,

$*$	e	φ	π	ϕ
e	e	φ	π	ϕ
φ	φ	π	ϕ	e
π	π	ϕ	e	φ
ϕ	ϕ	e	φ	π

$x * y \in L$ for any $x, y \in L$. Since $e^{-1} = e, \varphi^{-1} = \phi$, $\phi^{-1} = \varphi$ and $\pi^{-1} = \pi$, it follows that $x^{-1} \in L$ for any $x \in L$, so that by Two-step test L is a subgroup of G.

Let $M = \{e, \pi, h, v\}$. By the Cayley Table,

$*$	e	π	h	v
e	e	π	h	v
π	π	e	v	h
h	h	v	e	π
v	v	h	π	e

$x * y \in M$ for any $x, y \in M$. Since $x^{-1} = x$ for any $x \in M$, it follows that M is a subgroup of G.

Thus L and M are two distinct subgroups of order 4.

(e) (**4 marks**) Suppose G has a subgroup of order 5. Then by Lagrange's theorem, $5 \mid |G| = 8$, which is impossible.

3. (a) (i) (**3 marks**) Suppose a is a lower bound and b is an upper bound of A. Then $a \leq x \leq b$ for any $x \in A$. If $y \in B$, then $y = -v$ for some $v \in A$, so that $a \leq v \leq b$ and

$$-b \leq y = -v \leq -a.$$

Thus B is bounded below by $-b$ and above by $-a$.

(ii) (**5 marks**) Let U_A be the set of upper bounds of A. Then L is the least element of U_A. Let L_B be the set of lower bounds of B.

Then $a \in U_A \iff (\forall\, x \in A)(x \leq a) \iff (\forall\, x \in A)(-a \leq -x) \iff (\forall\, y \in B)(-a \leq y) \iff -a \in L_B$. In particular, $-L \in L_B$ and $-L \leq g := \text{glb } B$.

But $-g \in U_A$, so $L \leq -g$, $g \leq -L$ and $g = -L$.

(iii) (**5 marks**) Since $M \in L_B$, it follows that $M \leq y$ for any $y \in B$. If $x \in A$, then $y = -x \in B$, so that $M \leq -x$ and $x \leq -M$. Thus $-M \in U_A$.

For $\epsilon > 0$, there exists $y \in B$ such that $M \leq y \leq M + \epsilon$. Thus $y = -x$ for some $x \in A$ and

$$-M - \epsilon \leq x \leq -M.$$

So $-M = \text{lub } A$ and by (ii) above, $M = -(-M) = \text{glb } B$.

(b) Let $a = \text{glb } A$. For each $y \in Y$, there exists $x \in X$ such that $y = sx$. Thus $a \leq x$ and $sa \leq sx = y$ as $s > 0$, so sa is a lower bound of Y.

If $\epsilon > 0$, then $\frac{\epsilon}{s} > 0$, so there exists $x \in X$ such that $a \leq x \leq a + \frac{\epsilon}{s}$. Thus

$$sa \leq sx \leq s(a + \frac{\epsilon}{s}) = sa + \epsilon,$$

that is, $sa \leq y \leq sa + \epsilon$ for some $y = sx \in Y$. By (a) (iii) above,

$$sa = \text{glb } B.$$

4. (a) (**4 marks**) For each $n \in \mathbb{N}$, $n1_F \in F$. Since F is finite, there exist distinct integers $n, u \in \mathbb{N}$ such that

$$n1_F = u1_F.$$

If $n > u$, then $n1_F = (n-u)1_F + u1_F = u1_F + 0_F$ and by Cancellation low

$$m1_F = 0_F$$

with $m = n-u$. Similarly, if $n < u$, then $m1_F = 0_F$ with $m = u-n$. Thus there exists $m \in N$ such that $m1_F = 0_F$.

(b) (**4 marks**) Suppose p is the smallest $p \in N$ such that $p1_F = 0_F$, such p exists by (i) above. Suppose p is not a prime. Then $p = ab$ for some $a, b \in \mathbb{N}$ with $a \neq 1$ and $b \neq 1$. Then

$$(a1_F)(b1_F) = (ab)1_F = p1_F = 0_F,$$

so that $(a1_F) = 0_F$ or $(b1_F) = 0_F$, which is impossible since $a < p$ and $b < p$. It follows that p is a prime.

(c) (**3 marks**) For any $x \in F$,

$$px = x + x + \cdots + x = 1_F x + 1_F x + \cdots + 1_F x = (p1_F)x = 0_F x = 0_F.$$

5. (**7 marks**) For $\epsilon > 0$, let $N \in \mathbb{N}$ with $N > \frac{19+10\epsilon}{25\epsilon}$. If $n > N$, then

$$\left|\frac{2n+3}{5n-2} - \frac{2}{5}\right| = \left|\frac{19}{25n-10}\right| < \left|\frac{19}{25N-10}\right| \leq \frac{19}{25\frac{19+10\epsilon}{25\epsilon} - 10} = \epsilon.$$

Thus $\lim_{n\to\infty} \frac{2n+3}{5n-2} = \frac{2}{5}$.

6. (**15 marks**) Since $\lim_{n\to\infty} a_n = a$, it follows that for $\epsilon > 0$, there exists $N_1 > 0$ such that for all $n > N_1$,

$$|a_n - a| < \epsilon.$$

Similarly, since $\lim_{n\to\infty} b_n = a$, there exists $N_2 > 0$ such that for all $n > N_2$,

$$|b_n - a| < \epsilon.$$

Let $N = \max\{N_1, N_2\}$ and suppose $n > N$.

If $b_n \leq a_n$, then $c_n = a_n$, so that

$$|c_n - a| = |a_n - a| < \epsilon.$$

If $a_n < b_n$, then $c_n = b_n$, so that

$$|c_n - a| = |b_n - a| < \epsilon.$$

It follows that for all $n > N$,

$$|c_n - a| < \epsilon,$$

so that $\lim_{n\to\infty} c_n = a$.