1. (a) (**4 marks**) First we use Euclidean Algorithm to find gcd(946, 374):

n	x	y	
946	1	0	r_1
374	0	1	r_2
198	1	-2	$r_3 = r_1 - 2r_2$
176	-1	3	$r_4 = r_2 - r_3$
22	2	-5	$r_5 = r_3 - r_4$
0	-17	43	$r_6 = r_5 - 8r_4$

From this we see that $\gcd(946, 374) = 22$, and that $22 = 946 \cdot 2 + 374 \cdot (-5)$. Since 18 is not divisible by 22, it follows that $946x + 374y = 18$ has no integer solution.

(b) (**4 marks**) Use Euclidean Algorithm to find gcd(976, 3742):

n	y	x	
3742	1	0	r_1
976	0	1	r_2
814	1	-3	$r_3 = r_1 - 3r_2$
162	-1	4	$r_4 = r_2 - r_3$
4	6	-23	$r_5 = r_3 - 5r_4$
2	-241	924	$r_6 = r_4 - 40r_5$
0	$*$	$*$	$r_7 = r_5 - 2r_6$

From this we see that $\gcd(976, 3742) = 2$, and that $2 = 976 \cdot 924 + 3742 \cdot (-241)$.
Since $44 = 22 * 2$, it follows that

$$44 = 976 \cdot 20328 + 3742 \cdot (-5302)$$

and $(20328, -5302)$ is a solution. The general solution of the equation $976x + 3742y = 44$ is $x = 20328 - \frac{3742}{2}t = 20328 - 1871t$, $y = -5302 + \frac{976}{2}t = -5302 + 488t$ for $t \in \mathbb{Z}$.

(c) (**7 marks**) Use Euclidean Algorithm to find gcd(976, 374):

n	x	y	
976	1	0	r_1
374	0	1	r_2
228	1	-2	$r_3 = r_1 - r_2$
146	-1	3	$r_4 = r_2 - r_3$
82	2	-5	$r_5 = r_3 - r_4$
64	-3	8	$r_6 = r_4 - r_5$
18	5	-13	$r_7 = r_5 - 3r_6$
10	-18	47	$r_8 = r_6 - r_7$
8	23	-60	$r_9 = r_7 - r_8$
2	-41	107	$r_{10} = r_8 - r_9$
0	$*$	$*$	$r_{11} = r_9 - 4r_{10}$

From this we see that $\gcd(976, 374) = 2$, and that $2 = 976 \cdot (-41) + 374 \cdot 107$.
Since $22 = 11 * 2$, it follows that

$$22 = 976 \cdot (-451) + 374 \cdot 1177$$

and $(-451, 1177)$ is a solution. The general solution of the equation $976x + 374y = 22$ is $x = -451 - \frac{374}{2}t = -451 - 187t$, $y = 1177 + \frac{976}{2}t = 1177 + 488t$ for $t \in \mathbb{Z}$.
Now $0 \le -451 - 187t \le 40 \iff 451 \le -187t \le 491 \iff -\frac{491}{187} \le t \le -\frac{451}{187}$. Since there is no such t in $\mathbb{Z}$, it follows that $976x + 374y = 22$ has no solutions with $0 \le x \le 40$.

2. (a) (**5 marks**) $2x^2 - 3x - 4 \equiv 0 \pmod 5 \iff \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{0}$ in $\mathbb{Z}_5$. Now

$$\begin{aligned} \bar{x} = \bar{0} &\implies \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{1} \\ \bar{x} = \bar{1} &\implies \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{0} \\ \bar{x} = \bar{2} &\implies \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{3} \\ \bar{x} = \bar{3} &\implies \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{0} \\ \bar{x} = \bar{4} &\implies \bar{2}\bar{x}^2 + \bar{2}\bar{x} + \bar{1} = \bar{1}. \end{aligned}$$

Thus $\bar{x} = \bar{1}$ or $\bar{3}$ are the solutions in $\mathbb{Z}_5$, and so $x \in \bar{1} \cup \bar{3}$ are solutions, that is, $x \in \{5k+1, 5k+3 : k \in \mathbb{Z}\}$.

(b) (**7 marks**) $189x \equiv 28 \pmod{56} \iff 189x + 56y = 28$ for some $y \in \mathbb{Z} \iff 27x + 8y = 4$ for some $y \in \mathbb{Z} \iff 27x \equiv 4 \pmod 8 \iff \bar{3} \cdot_8 \bar{x} = \bar{4}$ in $\mathbb{Z}_8$.
Now

$\bar{x}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{3} \cdot_8 \bar{x}$	$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\bar{2}$	$\bar{5}$

Thus $3x \equiv 4 \pmod 8 \iff \bar{x} = \bar{4} \iff x \in \bar{4}$, that is, $x \in \{8k + 4 : k \in \mathbb{Z}\}$.

(c) (**8 marks**)

$$946x \equiv 26 \pmod{2316} \iff (\exists y \in \mathbb{Z})(946x + 2316y = 26) \iff (\exists y \in \mathbb{Z})(473x + 1158y = 13).$$

First we use Euclidean Algorithm to find $\gcd(473, 1158)$:

n	y	s	
1158	1	0	r_1
473	0	1	r_2
212	1	-2	$r_3 = r_1 - 2r_2$
49	-2	5	$r_4 = r_2 - 2r_3$
16	9	-22	$r_5 = r_3 - 2r_4$
1	-29	71	$r_6 = r_5 - 3r_4$

From this we see that $\gcd(1158, 473) = 1$, and that $1158 \cdot (-29) + 473 \cdot (71) = 1$.
Thus $13 = 1158 \cdot (-377) + 473 \cdot (923)$ and $x = 923 - \frac{1158}{1}t = 923 - 1158t$ for any $t \in \mathbb{Z}$. Now $x > 0 \iff 923 - 1158t > 0 \iff t < \frac{923}{1158} < 1$, so $t = 0$ and $x = 923$ is the smallest positive solution in $\mathbb{Z}$.

3. (**8 marks**) $14 \mid 21(15n+27)(n+28) \iff 21(15n+27)(n+28) \equiv 0 \pmod{14}$. Now

$$21(15n+27)(n+28) \equiv 7(n+13)n \equiv 7n(n-1) \pmod{14}.$$

If $n = 2m$, then $7n(n-1) = 14m(2m-1) \equiv 0 \pmod{14}$.

If $n = 2m+1$, then $7n(n-1) = 14(2m+1)m \equiv 0 \pmod{14}$.

Thus $21(15n+27)(n+28) \equiv 0 \pmod{14}$ for all $n \in \mathbb{N}$ and $14 \mid 21(15n+27)(n+28)$.

4. (a) (**5 marks**) We first divide $b(x)$ into $a(x)$, then divide the remainder into $b(x)$, and so on, until we get a remainder of 0.

$$\begin{array}{r|rrrr}
 & & & & 1 \\ \hline
x^3-2x-1 \;) & x^3 & +5x^2 & +2x & -2 \\
 & x^3 & & -2x & -1 \\ \hline
 & & 5x^2 & +4x & -1
\end{array}$$

so $a(x) = b(x) + 5x^2 + 4x - 1$, and then

$$\begin{array}{r|rrrr}
 & & & \frac{1}{5}x & -\frac{4}{25} \\ \hline
5x^2+4x-1 \;) & x^3 & & -2x & -1 \\
 & x^3 & +\frac{4}{5}x^2 & -\frac{1}{5}x & \\ \hline
 & -\frac{4}{5}x^2 & -\frac{9}{5}x & -1 & \\
 & -\frac{4}{5}x^2 & -\frac{16}{25}x & +\frac{4}{25} & \\ \hline
 & & -\frac{29}{25}x & -\frac{29}{25} &
\end{array}$$

But now it is easy to see $x+1$ is a factor of $5x^2+4x-1$ since $5 \cdot (-1)^2 + 4 \cdot (-1) - 1 = 0$. Hence factorizing $5x^2+4x-1 = (x+1)(5x-1)$. Thus the greatest monic common divisor is

$$\gcd(a(x), b(x)) = \gcd(b(x), 5x^2+4x-1) = \gcd(5x^2+4x-1, -\frac{29}{25}x - \frac{29}{25}) = x+1.$$

(b) (i) Using long division in $\mathbb{Z}_5[x]$ we have

$$\begin{array}{r|rrrrr}
 & & & & 2x & \\ \hline
3x^3+x^2+x+2 \;) & x^4 & +2x^3 & & +4x & +1 \\
 & x^4 & +2x^3 & +2x^2 & +4x & \\ \hline
 & & & 3x^2 & & +1
\end{array}$$

Thus

$$x^4 + 2x^3 + 4x + 1 = (3x^3 + x^2 + x + 2)(2x) + (3x^2 + 1),$$

so that $q(x) = 2x$ and $r(x) = 3x^2 + 1$.

(ii) (**6 marks**) Using long division again, we have

$$\begin{array}{r|rrrr}
 & & & x & +2 \\ \hline
3x^2+1 \;) & 3x^3 & +x^2 & +x & +2 \\
 & 3x^3 & & x & \\ \hline
 & & x^2 & & +2 \\
 & & x^2 & & +2 \\ \hline
 & & & & 0
\end{array}$$

Thus

$$3x^3 + x^2 + x + 2 = (3x^2 + 1)(x + 2) + 0.$$

It follows that $3x^2+1$ is a gcd and $2(3x^2+1) = x^2+2$ is the monic $\gcd(f(x), g(x))$. Now

$$3x^2 + 1 = (x^4 + 2x^3 + 4x + 1) - (3x^3 + x^2 + x + 2)(2x),$$

so that

$$x^2 + 2 = 2(x^4 + 2x^3 + 4x + 1) + (3x^3 + x^2 + x + 2)(x).$$

Thus $u(x) = 2$ and $v(x) = x$.

5. (a) (**4 marks**) If $a * b = c * b$, then $a = a * e = a * (b * b^{-1}) = (a * b) * b^{-1} = (c * b) * b^{-1} = c * (b * b^{-1}) = c * e = c$.

(b) (**4 marks**) If $a * b = e$, then $a * (b * a) = (a * b) * a = e * a = a = a * e$, so by Cancellation, $b * a = e$.

6. (a) (**9 marks**) For any $x, y \in A$, $x * y = 3xy \in \mathbb{R}$ and $3xy \neq 0$, so that $*$ is a binary operation on A.

$x * (y * z) = x * (3yz) = 3x(3yz) = 9xyz$ and $(x * y) * z = 3(x * y)z = 3(3xy)z = 9xyz$. Thus $x * (y * z) = (x * y) * z$.

Since $x * y = 3xy = 3yx = y * x$, $*$ is commutative.

If $e = \frac{1}{3}$, then $x * e = 3xe = x$ for all $x \in A$ and so e is the identity.

For $a \in A$, let $b = \frac{1}{9a}$. Then $b \in A$ and $a * b = 3ab = e$ and b is the inverse of a.

It follows that $(A, *)$ is an abelian group.

(b) (**3 marks**) Take $x = y = \sqrt{2}$, so that $x, y \in T$. But $x * y = 3xy = 6 \notin T$, so $*$ is not a binary operation on T.