1. (**2 marks each**)

(a) (i) $-7 = 5(-2) + 3$, and $7 = 5 + 2$, so $f(-7) = 3$ and $f(7) = 2$.

(ii) f is not one-to-one because $f(3) = f(8) = 3$ with $3, 8 \in A$.

(iii) f is not onto because $f(x) \neq -1$ for all $x \in A$.

(b) (i) $g(\{-1, 0, 1\}) = \{a \in A : (f(a) = 0) \vee (f(a) = 1)\} = \{-5, 0, 5, -4, 1, 6\}$.

(ii) g is not one-to-one because $g(\{-3\}) = g(\{-7\}) = \emptyset$ with $\{-3\}, \{-7\} \in \mathcal{P}(A)$.

(iii) g is not onto because $g(X) \neq \{-1\}$ for all $X \in \mathcal{P}(A)$.

(c) (i) $\sim$ is reflexive since $f(a) = f(a)$ for all $a \in A$, symmetric since $a \sim b \iff f(a) = f(b) \iff f(b) = f(a) \iff b \sim a$, and transitive since $(a \sim b) \wedge (b \sim c) \iff f(a) = f(b) = f(c) \implies f(a) = f(c) \iff a \sim c$.

(ii) For $a \in A$, $a \in S \iff a \sim 7 \iff f(a) = f(7) = 2 \iff a = 5q + 2$ for some $q \in \mathbb{Z} \iff a - 2 = 5q$ for some $q \in \mathbb{Z}$. Thus

$$S = \{-8, -3, 2, 7\}.$$

(iii) $[0] = \{-5, 0, 5\}$, $[1] = \{-9, -4, 1, 6\}$, $[2] = \{-8, -3, 2, 7\}$, $[3] = \{-7, -2, 3, 8\}$ and $[4] = \{-6, -1, 4, 9\}$.

2. (a) (**6 marks**) Suppose $a, a' \in A$, Then

$$\begin{aligned} f(a) = f(a') &\iff \frac{5a}{a-3} = \frac{5a'}{a'-3} \\ &\iff \frac{5(a-3)+15}{a-3} = \frac{5(a'-3)+15}{a'-3} \\ &\iff 5 + \frac{15}{a-3} = 5 + \frac{15}{a'-3} \\ &\iff \frac{15}{a-3} = \frac{15}{a'-3} \\ &\iff a' - 3 = a - 3 \\ &\iff a' = a, \end{aligned}$$

so f is one-one.

Now we show that f is onto.

For any $b \in B$, let $a = \frac{3b}{b-5}$. Since $b \neq 5$, it follows that $a \in \mathbb{R}$. If $a \notin A$, then $a = 3$, i.e. $b = b - 5$ or $0 = -5$, which is impossible. Thus $a \in A$ and

$$f(a) = \frac{5\frac{3b}{b-5}}{\frac{3b}{b-5} - 3} = \frac{15b}{3b - (3b - 15)} = b,$$

so f is onto.

(b) (**2 marks**) As shown above $f^{-1} : B \to A$ is given by $f^{-1}(x) = \frac{3x}{x-5}$

3. (a) (**6 marks**) Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_- = \{-1, -2, -3, \ldots\}$, $E = \{2, 4, 6, \ldots\}$ and $O = \{1, 3, 5, \ldots\}$. Then $\mathbb{Z}_+ \cap \mathbb{Z}_- = E \cap O = \emptyset$, $\mathbb{Z} = \mathbb{Z}_+ \cup \mathbb{Z}_-$ and $\mathbb{N} = E \cup O$.

If $x \in \mathbb{Z}_+$, then $f(x) = 2x + 2 \geq 2$, so that $f(x) \in E$. If $x \in \mathbb{Z}_-$, then $f(x) = -2x - 1 \geq 1$, so that $f(x) \in O$. Thus f can be viewed as a function from $\mathbb{Z}_+$ to E and from $\mathbb{Z}_-$ to O.

Let $a, a' \in \mathbb{Z}$ such that $f(a) = f(a')$. If $f(a)$ is even, then so is $f(a')$ and so $a, a' \in \mathbb{Z}_+$ as $f(x)$ is odd for $x \in \mathbb{Z}_-$. Thus $2a + 2 = 2a' + 2$ and $a = a'$.

If $f(a)$ is odd, then so is $f(a')$ and so $a, a' \in \mathbb{Z}_-$ as $f(x)$ is even for $x \in \mathbb{Z}_+$. Thus $-2a - 1 = -2a' - 1$ and $a = a'$. So f is one-one.

Let $b \in \mathbb{N}$. If $b = 2m \in E$ for some $m \in \mathbb{N}$, then $a = m - 1 \geq 0$ and so $a \in \mathbb{Z}_+$ with $f(a) = 2a + 2 = 2(m - 1) + 2 = 2m = b$. If $b = 2m - 1 \in O$ for some $m \in \mathbb{N}$, then $a = -m \in \mathbb{Z}_-$ and $f(a) = -2a - 1 = 2m - 1 = b$. So f is onto.

(b) (**3 marks**) For $x \in \mathbb{Z}$, define a function g by

$$
\begin{aligned}
g(x) &= 2x + 2 \quad \text{if} \quad x \geq 0 \\
&= -2x + 1 \quad \text{if} \quad x < 0.
\end{aligned}
$$

Then g is a one-to-one function from $\mathbb{Z}$ to $\mathbb{N}$. Claim that g is not onto. Take $b = 1 \in \mathbb{N}$. If $g(x) = b$ for some $x \in \mathbb{Z}$, then $x \in \mathbb{Z}_-$ as b is odd. Thus $-2x + 1 = 1$ and $x = 0 \in \mathbb{Z}_+$, which is impossible.

(c) (**3 marks**) For $x \in \mathbb{Z}$, define a function h by

$$
\begin{aligned}
h(x) &= x + 1 \quad \text{if} \quad x \geq 0 \\
&= -x \quad \text{if} \quad x < 0.
\end{aligned}
$$

Then h is a function from $\mathbb{Z}$ to $\mathbb{N}$. Since $h(-1) = h(0) = 1$, it is not one-one.

(d) (**2 marks**) For $x \in \mathbb{Z}$, define a function $k : \mathbb{Z} \to \mathbb{N}$ by $k(x) = 1$. Then k is neither one-one nor onto.

4. (**5 marks each**)

(a) Define $f : A \to \mathbb{N}$ by $f(-\frac{1}{n}) = n$. Then f is a function.

f **is onto**. For any $b \in \mathbb{N}$, set $a = -\frac{1}{b}$. Then $a \in A$ and $f(a) = b$.

f **is strictly order preserving**. Let $a = -\frac{1}{n}$ and $a' = -\frac{1}{n'}$ be two elements of A for some $n, n' \in \mathbb{N}$. Then

$$
\begin{aligned}
a \leq a' &\iff -\frac{1}{n} \leq -\frac{1}{n'} \\
&\iff \frac{1}{n'} \leq \frac{1}{n} \\
&\iff n \leq n' \\
&\iff f(a) \leq f(a').
\end{aligned}
$$

Thus f is an order isomorphism and so $A \simeq \mathbb{N}$.

(b) Note that $B = \{-\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\} = \{\ldots, -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ and

$$1 > \frac{1}{2} > \frac{1}{3} > \ldots > -\frac{1}{4} > -\frac{1}{3} > -\frac{1}{2} > -1.$$

In particular, -1 is the least element of B.

Suppose $B \simeq \mathbb{Z} \setminus \{0\}$ and f is an order isomorphism from B onto $\mathbb{Z} \setminus \{0\}$. Then $f(-1) = m \in \mathbb{Z} \setminus \{0\}$ is also the least element. Since $-1 \in \mathbb{Z} \setminus \{0\}$ and m is the least element, it follows that $m \leq -1$ and so $m - 1 \in \mathbb{Z} \setminus \{0\}$. But $m - 1 < m$, so m is not a least element. Contradiction.

5. (**5 marks each**)

(a) Since $d = \gcd(a, b)$, there exist $x, y \in \mathbb{Z}$ such that

$$ax + by = d.$$

Since $a = da_1$ and $b = db_1$, it follows that $d = da_1x + db_1y$, so that $1 = a_1x + b_1y \iff \gcd(a_1, b_1) = 1$.

(b) Suppose $d = \gcd(p, a)$. Then $d|p$ and so $d = p$ or $d = 1$. If $d \neq 1$, then $d = p$ and hence $p|a$.

(c) Let $w = 2m + 1$ for some $m \in \mathbb{Z}$. Then

$$\gcd(w, w + 2) = \gcd(w, 2) = \gcd(2m + 1, 2) = \gcd(1, 2) = 1.$$

OR

Let $d = \gcd(w, w + 2)$, so that $d|w$ and $d|(w + 2)$. Thus $d|2 = (w + 2) - w$, and $d = 1$ or 2. If $d = 2$, then $2|w$ and w is even. A contradiction. So $d = 1$ and $w, w + 2$ are coprime.

(d) Let $d = \gcd(a, b)$, so that $d = ax + by$ for some $x, y \in \mathbb{Z}$. Now

$$d = ax + by = ax + aky + by - aky = a(x + kb) + b(y - ka)$$

for any $k \in \mathbb{Z}$. It follows that there are infinitely many $s, t \in \mathbb{Z}$ such that $d = as + bt$.

6. (**3 marks each**)

(a) The algorithm gives us

n	x	y	
288	1	0	r_1
51	0	1	r_2
33	1	-5	$r_3 = r_1 - 5r_2$
18	-1	6	$r_4 = r_2 - r_3$
15	2	-11	$r_5 = r_3 - r_4$
3	-3	17	$r_6 = r_4 - r_5$
0	*	*	$r_7 = r_5 - 5r_6$

From which we see that $\gcd(288, 51) = 3$ and $3 = 288 \cdot (-3) + 51 \cdot 17$.

(b) The algorithm gives us

n	x	y	
629	1	0	r_1
357	0	1	r_2
272	1	-1	$r_3 = r_1 - r_2$
85	-1	2	$r_4 = r_2 - r_3$
17	4	-7	$r_5 = r_3 - 3r_4$
0	$*$	$*$	$r_6 = r_4 - 5r_5$

From this we see that $\gcd(629, 357) = 17 = 629 \cdot (4) + 357 \cdot (-7)$.

(c) The algorithm gives us

n	x	y	
252	1	0	r_1
180	0	1	r_2
72	1	-1	$r_3 = r_1 - r_2$
36	-2	3	$r_4 = r_2 - 2r_3$
0	$*$	$*$	$r_5 = r_3 - 2r_4$

from which we see that $\gcd(252, 180) = 36 = 252 \cdot (-2) + 180 \cdot (3)$.