1. (a) (**3 marks**) Let $E(x) :=$ “x is even. Then

$$(a) \iff (\forall a, b, c \in \mathbb{Z})(E(ac) \wedge E(ab) \wedge E(bc) \implies E(a) \wedge E(b) \wedge E(c)).$$

$$\neg(a) \iff (\exists a, b, c \in \mathbb{Z})(E(ac) \wedge E(ab) \wedge E(bc) \wedge (O(a) \vee O(b) \vee O(c))).$$

(1) Take $a = 2 = b$ and $c = 1$.
(2) Then $ac = 2 = bc$ and $ab = 4$ are even but c is odd.
Thus $a = 2, b = 2$ and $c = 1$ is a counterexample to the statement (a).

(b) (**3 marks**) Let $O(x) :=$ “x is odd. Then

$$(b) \iff (\exists xy \in \mathbb{Z})(E(xy) \wedge O(x) \wedge O(y)).$$

$$\neg(b) \iff (\forall x, y \in \mathbb{Z})(E(xy) \implies (E(x) \vee E(y))).$$

Suppose, for a contradiction that x and y are both odd but xy is even for some $x, y \in \mathbb{Z}$. Then $x = 2k + 1$ and $y = 2t + 1$ for some $k, t \in \mathbb{Z}$. Thus

$$xy = (2k + 1)(2t + 1) = 4kt + 2k + 2t + 1 = 2(2kt + k + t) + 1.$$

Since $2kt + k + t$ is an integer, it follows that xy is odd, which is impossible.

2. (**1 mark**) For $n \in \mathbb{N}$, let P_n be the statement that $7 \mid (4^{2n} - 2^n)$.

(**2 mark**)

Base case: When $n = 1$, we have $4^{2n} - 2^n = 16 - 2 = 14 = 7 \times 2$, so $7 \mid (4^{2n} - 2^n)$.

(**4 mark**)

Inductive step: Let $k \in \mathbb{N}$ and suppose P_k is true, that is, $4^{2k} - 2^k = 7m$, for some integer m. Then $4^{2k} = 7m + 2^k$ and

$$\begin{aligned}
4^{2(k+1)} - 2^{k+1} &= 4^{2k+2} - 2^{k+1} \\
&= 16 \times 4^{2k} - 2 \times 2^k \\
&= 16 \times (7m + 2^k) - 2 \times 2^k \\
&= 16 \times 7m + 14 \times 2^k \\
&= 7(16m + 2 \times 2^k),
\end{aligned}$$

so it is divisible by 7. It follows that P_{k+1} is true and by mathematical induction, P_n is true for all $n \in \mathbb{N}$.

3. (**1 mark**) For $n \in \mathbb{N}$ with $n \geq 4$, let P_n be the statement that $n! > 2^n$.

(**2 mark**)

Base case: When $n = 4$, we have $4! = 24$ and $2^n = 16$, so $n! > 2^n$, that is P_4 is true.

(**4 mark**)

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 4$ and suppose P_k is true, that is, $k! > 2^k$. Then

$$\begin{aligned} (k+1)! &= (k+1) \times k! \\ &> (k+1) \times 2^k \qquad \text{as} \qquad k! > 2^k \\ &> 2 \times 2^k \qquad \text{as} \qquad k+1 > 4 \\ &= 2^{k+1}, \end{aligned}$$

so P_{k+1} is true. By induction, P_n is true for all $n \in \mathbb{N}$ with $n \geq 4$.

4. (**3 marks**) For $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ let P_n be the statement that x^n is even if and only if x is even.

Base: P_1 is the statement that x is even if and only if x is even, which is clearly true.

(**5 marks**) [Inductive step:] Suppose $k \in \mathbb{N}$ and suppose P_k is true, in other words x^k is even if and only if x is even.

Suppose x is even, so that $x = 2t$ for some $t \in \mathbb{Z}$. So $x^{k+1} = x \cdot x^k = 2(t2^k)$ is even.

Suppose x^{k+1} is even. From Question 1 (b) of this assignment, we know that for $a, b \in \mathbb{Z}$ if ab is even then either a or b is even. Now $x^{k+1} = x \cdot x^k$ is even, so x is even or x^k is even. If x is even, then P_{k+1} is true. On the other hand, if x^k is even, then by the induction hypothesis, x is even as well. Thus P_{k+1} is always true. Hence, by complete induction, P_n is true for all $n \in \mathbb{N}$.

5. (**2 marks**) Compute $x_1 = 3, x_2 = 18 = 2 \times 3^2, x_3 = 3 \times 3^3, x_4 = 4 \times 3^4$. We conjecture that

$$x_n = n3^n.$$

(**2 marks**) For $n \in \mathbb{N}$ let P_n be the statement that $x_n = n3^n$.

Base: P_1 is the statement that $x_1 = 3$, which is true; P_2 is the statement that $x_2 = 2 \cdot 3^2 = 18$, which is also true.

(**4 marks**) [Inductive step:] Suppose $k \in \mathbb{N}$ with $k \geq 2$ and suppose $P_1 \wedge \ldots \wedge P_k$ is true, in other words $x_i = i3^i$ for $1 \leq i \leq k$ Then

$$\begin{aligned} x_{k+1} &= 6x_k - 9x_{k-1} \\ &= 6k3^k - 9(k-1)3^{k-1} \\ &= 2k3^{k+1} - (k-1)3^{k+1} \\ &= (2k - k + 1)3^{k+1} \\ &= (k+1)3^{k+1}, \end{aligned}$$

so $x_{k+1} = (k+1)3^{k+1}$, in other words P_{k+1} is true.

Hence, by complete induction, P_n is true for all $n \in \mathbb{N}$.

6. (**3 marks each**)

(a) Note that $\rho = \{(x, y) \in A \times A : 2x + y = 0\} = \emptyset$ since both $x < 0$ and $y < 0$. So ρ is **symmetric, antisymmetric** and **transitive**. But ρ is **not reflexive** because $(-1, -1) \notin \rho$.

(b) **Not reflexive**: $(0, 0) \notin \rho$.
Symmetric: $(x, y) \in \rho \iff x + y = 1 \iff y + x = 1 \iff (y, x) \in \rho$;
Not antisymmetric: $0\rho 1 \wedge 1\rho 0$ but $1 \neq 0$.
Not transitive: $0\rho 1 \wedge 1\rho 0$ but $(0, 0) \notin \rho$.

(c) **Not reflexive**: $(b, b) \notin \rho$.
Not symmetric: $(a, c) \in \rho$ but $(c, a) \notin \rho$.
Not antisymmetric: $(a, b) \in \rho \wedge (b, c) \in \rho$ but $a \neq c$.
Not transitive: $(b, a) \in \rho \wedge (a, b) \in \rho$ but $(b, b) \notin \rho$.

(d) **Reflexive**: for all $x \in D$, $|x - x| = 0 < 2$.
Symmetric: $(x, y) \in \rho \iff |x - y| < 2 \iff |y - x| < 2 \iff (y, x) \in \rho$.
Not antisymmetric: $1\rho 0 \wedge 0\rho 1$ but $0 \neq 1$.
Not transitive: $1\rho 2 \wedge 2\rho 3$, that is, $|1 - 2| < 2$ and $|2 - 3| < 2$, but $|x - z| = |1 - 3| = 2 \not< 2$, namely $x \not\rho z$.

7. (a) (**2 marks**) Let $Q = A \times A$. It is easy to check that Q is an equivalents relation and $R \subseteq Q$.

(b) (**5 marks**) For any $x \in A$, if $Q \in \Omega$, then $(x, x) \in Q$ since Q is reflexive. Thus $(x, x) \in S$ and so S is **reflexive**.
If $(x, y) \in S$, then $(x, y) \in Q$ for any $Q \in \Omega$, so that $(y, x) \in Q$ as Q is symmetric. Thus $(y, x) \in S$ and so S is **symmetric**.
If $(x, y) \in S$ and $(y, z) \in S$, then $(x, y) \in Q$ and $(y, z) \in Q$ for any $Q \in \Omega$, so that $(x, z) \in Q$ as Q is transitive. Thus $(x, z) \in S$ and so S is **transitive**.
It follows that S is an equivalence relation.
For any $(x, y) \in R$ and any $Q \in \Omega, (x, y) \in Q$ as $R \subseteq Q$. Thus $(x, y) \in S$ and $R \subseteq S$.

(c) (**3 marks**) If X is an equivalence relation containing R. Then $X \in \Omega$. For any $(x, y) \in S$, $(x, y) \in Q$ for any $Q \in \Omega$. In particular, $(x, y) \in X$ and so $S \subseteq X$.

8. (a) (**1 mark**) $\sim$ is reflexive because $8 \mid 3x + 5x = 8x$ for any $x \in \mathbb{Z}$.
(**3 marks**) $\sim$ is symmetric. Suppose $x \sim y$. Then $8 \mid 3x + 5y$, so $3x + 5y = 8m$ for some $m \in \mathbb{Z}$, and

$$8 \mid 3y + 5x = 8x + 8y - (3x + 5y) = 8(x + y - m).$$

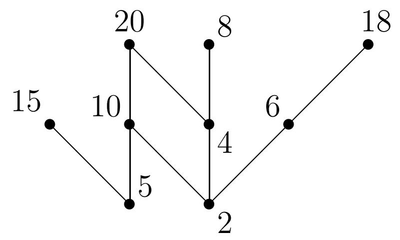
Thus $y \sim x$.
(**3 marks**) $\sim$ is transitive. Suppose $x \sim y$ and $y \sim z$ for $x, y, z \in \mathbb{Z}$. Then $8 \mid 3x + 5y$ and $8 \mid 3y + 5z$, so $8 \mid 3x + 5z = (3x + 5y) + (3y + 5z) - 8y$ and $x \sim z$.

(b) (**3 marks**)

$$x \in [0] \iff 8 \mid 5x + 3 \cdot 0 = 5x \iff 8|x.$$

Thus $[0] = \{x \in \mathbb{Z} : x = 8t, \exists\, t \in \mathbb{Z}\} = 8\mathbb{Z}$.

9. (a) (**3 marks**) We have the lattice diagrams



(b) (**2 marks**) maximal elements = $\{15, 20, 8, 18\}$; minimal elements = $\{5, 2\}$.

(c) (**3 marks**) Let $S = \{15, 18\}$. Then S has no upper bound and no lower bound.

(d) (**2 marks**) 2 is the greatest lower bound of $\{4, 6, 10\}$.

(e) (**2 marks**) $\{2, 4, 20\}$ is totally ordered, since $2 \mid 4 \mid 20$. Thus it is totally ordered.