1. (**2 marks each**)

(a) "Every even integer is a multiple of 4" is a statement. If $E(x) =$ "x is an even number" and $M(x) =$ "x is a multiple of 4", then the statement (a) can be translated as $(\forall n \in \mathbb{Z})(E(n) \implies M(n))$.

(b) "If n is a prime number then n^2 is not even" is a statement, which can be translated as $(\forall n \in \mathbb{Z})(P(n) \implies \sim E(n^2))$, where $P(x) =$ "x is a prime number".

(c) "x^2 is positive" is a predicate (with x a free variable), which can be translated as $W(x)$, where $W(x) =$ "x is positive".

(d) "Find an even number" is neither a statement nor a predicate. It is a command.

(e) "For any integer n there is an even number m such that $m + n = -n$" is a statement, which can be translated as $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})(E(m) \wedge S(m,n))$, where $S(x,y) =$ "$x + y = -y$".

2. (**3 marks each**)

(a) We have the following truth table:

A	B	$\sim B$	$\sim A$	$A \vee \sim B$	$\sim A \wedge B$	$(A \vee \sim B) \wedge (\sim A \wedge B$
T	T	F	F	T	F	F
T	F	T	F	T	F	F
F	T	F	T	F	T	F
F	F	T	T	T	F	F

Since the last column contains only "F"s, $(A \vee \sim B) \wedge (\sim A \wedge B)$ is a contradiction.

(b) We have the following truth table:

A	B	$\sim A$	$\sim B$	$A \implies \sim B$	$(A \implies \sim B) \implies \sim A$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	T

Since the last column contains both "T" and "F", $(A \implies \sim B) \implies \sim A$ is neither a tautology nor a contradiction.

(c) We have the following truth table:

A	B	$\sim B$	$\sim A$	$\sim A \implies B$	$\sim B \wedge A$	$(\sim A \implies B) \implies (\sim B \wedge A)$
T	T	F	F	T	F	F
T	F	T	F	T	T	T
F	T	F	T	T	F	F
F	F	T	T	F	F	F

Since the last column contains both "T" and "F", $(A \implies \sim B) \implies (\sim B \implies A)$ is neither a tautology nor a contradiction.

(d) We have the following truth table:

A	B	$\sim A$	$\sim B$	$B \vee \sim B$	$(\sim A) \wedge (B \vee \sim B)$	$(\sim A) \iff (\sim A) \wedge (B \vee \sim B)$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since the last column contains only "T"s, $(\sim A) \iff (\sim A) \vee (\sim B \wedge B)$ is a tautology.

3. (a) (**3 marks**) The negation of $A(n)$ is "$n = 3q - 1$ or $n = 3q - 2$ for some $q \in \mathbb{Z}$ and $n^2 \neq 3k + 1$ for all $k \in \mathbb{Z}$."

(b) (**4 marks**) The contrapositive of $A(n)$ is "if $n^2 \neq 3k + 1$ for all $k \in \mathbb{Z}$, then $n \neq 3q - 1$ and $n \neq 3q - 2$ for any $q \in \mathbb{Z}$."

(c) (**3 marks**) The converse of $A(n)$ is "if $n^2 = 3k+1$ for some $k \in \mathbb{Z}$, then $n = 3q-1$ or $n = 3q-2$ for some $q \in \mathbb{Z}$."

(d) (**5 marks**) Suppose $n = 3q - 1$ or $n = 3q - 2$ for some $q \in \mathbb{Z}$. If $n = 3q - 1$, then $n^2 = 9q^2 - 6q + 1 = 3(3q^2 - 2q) + 1 = 3k + 1$ with $k = (3q^2 - 2q) \in \mathbb{Z}$.
If $n = 3q-2$, then $n^2 = 9q^2 - 12q + 4 = 3(3q^2 - 4q + 1) + 1 = 3k + 1$ with $k = (3q^2 - 4q + 1) \in \mathbb{Z}$.
Thus $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$.

(e) (**3 marks**) By (d), $A(n)$ is true for all $n \in \mathbb{Z}$, so $A(n)$ is true for all $n \in \mathbb{N}$. The contrapositive of $A(n)$ is true for all $n \in \mathbb{N}$ because $A(n)$ is true for all $n \in \mathbb{N}$ and the contrapositive of $A(n)$ is equivalent to $A(n)$.

(f) (**5 marks**) Suppose, for a contradiction that $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$ but $n \neq 3q - 1$ and $n \neq 3q - 2$ for any $q \in \mathbb{Z}$. Then $n = 3q$ for some $q \in \mathbb{Z}$ and $n^2 = 9q^2 = 3k + 1$, so that $3 \mid 3(3q^2 - k) = 9q^2 - 3k = 1$. A contradiction.

4. (**5 marks each**)

(a) Suppose $A \subseteq B$. Then $a \in A \implies a \in B$, so

$$\begin{aligned}(a, c) \in A \times C &\iff (a \in A) \wedge (c \in C)\\ &\implies (a \in B) \wedge (c \in C)\\ &\iff (a, c) \in B \times C.\end{aligned}$$

Thus $(a, c) \in A \times C \implies (a, c) \in B \times C$, i.e. $A \times C \subseteq B \times C$.
Conversely, suppose $A \times C \subseteq B \times C$, so that $(a, c) \in A \times C \implies (a, c) \in B \times C$. Since $C \neq \emptyset$, it follows that there is an element $c \in C$. If $a \in A$, then $(a, c) \in A \times C$ and so $(a, c) \in B \times C$ as $A \times C \subseteq B \times C$. Thus $a \in B$ (and $c \in C$) and so $a \in A \implies a \in B$, i.e. $A \subseteq B$.

(b)

$$\begin{aligned}(a, b) \in A \times (B \cap C) &\iff (a \in A) \wedge (b \in (B \cap C))\\ &\iff (a \in A) \wedge ((b \in B) \wedge (b \in C))\\ &\iff ((a \in A) \wedge (b \in B)) \wedge ((a \in A) \wedge (b \in C))\\ &\iff ((a, b) \in A \times B) \wedge ((a, b) \in A \times C)\\ &\iff (a, b) \in (A \times B) \cap (A \times C),\end{aligned}$$

so $(a,b) \in A\times(B\cap C) \iff (a,b) \in (A\times B)\cap(A\times C)$, and hence $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

(c)

$$\begin{aligned}(a,b) \in (A \times B) \cap (C \times D) &\iff ((a,b) \in A \times B) \wedge ((a,b) \in C \times D)\\ &\iff ((a \in A) \wedge (b \in B)) \wedge ((a \in C) \wedge (b \in D))\\ &\iff ((a \in A) \wedge (a \in C)) \wedge ((b \in B) \wedge (b \in D)\\ &\iff ((a \in A \cap C) \wedge (b \in B \cap D))\\ &\iff (a,b) \in (A \cap C) \times (B \cap D),\end{aligned}$$

so $(a,b) \in (A \times B) \cap (C \times D) \iff (a,b) \in (A \cap C) \times (B \cap D)$, and hence $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(d)

$$\begin{aligned}(a,b) \in (A \times B) \cup (C \times D) &\iff ((a,b) \in A \times B) \vee ((a,b) \in C \times D)\\ &\iff ((a \in A) \wedge (b \in B)) \vee ((a \in C) \wedge (b \in D))\\ &\implies ((a \in A) \vee (a \in C)) \wedge ((b \in B) \vee (b \in D))\\ &\iff ((a \in A \cup C) \wedge (b \in B \cup D))\\ &\iff (a,b) \in (A \cup C) \times (B \cup D),\end{aligned}$$

so $(a,b) \in (A \times B) \cap (C \times D) \implies (a,b) \in (A \cup C) \times (B \cup D)$, and hence $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

To show that $(A \times B) \cap (C \times D) \neq (A \cap C) \times (B \cap D)$ in general, we give an example.

Let $A = \{1\} = D$ and $B = C = \emptyset$. Then $A \times B = C \times D = \emptyset$ and so $(A \times B) \cup (C \times D) = \emptyset$. But $A \cup C = B \cup D = \{1\}$, so $(A \cup C) \times (B \cup D) = \{(1,1)\} \neq \emptyset$.

5. (**3 marks each**)

(a)

$$\mathcal{P}(Y) = \{\emptyset, \{a\}, \{b\}, \{9\}, \{a,b\}, \{a,9\}, \{b,9\}, \{a,b,9\}\},$$

$X \cap Y = \{9\}$, and $\mathcal{P}(X \cap Y) = \{\emptyset, \{9\}\}$.

(b) $X \cup Y = \{1, 9, a, b\}$, and

$$\begin{aligned}\mathcal{P}(X \cup Y) = \{&\emptyset, \{1\}, \{9\}, \{a\}, \{b\}, \{1,9\}, \{1,a\}, \{1,b\}, \{9,a\},\\ &\{9,B\}, \{a,b\}, \{1,9,a\}, \{1,9,b\}, \{1,a,b\}, \{9,a,b\}, \{1,5,9,a,b\}\}.\end{aligned}$$

6. (a) (**2 marks**) $S = \{1,2,3,4\} \in \mathcal{P}(A)$ and the number of elements of S is 4.

(b) (**3 marks**) $S = \{\emptyset, \{1\}, \{3\}, A\} \subseteq \mathcal{P}(A)$ and the number of elements of S is 4.

(c) (**4 marks**) Let $S = \{\emptyset, \{1\}, \{3\}, \{1,3,5\}\}$ and $B = \{1,3,5\}$. Then $B \in S$, $S \subseteq \mathcal{P}(A)$, B has 3 elements and S has 4 elements.