1. (a) Since $x * y \in H$ for any $x, y \in H$, it follows that $x^u \in H$ for any $u \in \mathbb{N}$.
If for any $m, n \in \mathbb{N}$ with $m \neq n$, $x^m \neq x^n$, then H has a subset $\{x^u : u \in \mathbb{N}\}$ containing infinity many elements, so that H is not finite, which is impossible. Thus $x^m = x^n$ for some $m \neq n$.

(b) By Two-step test, it suffices to show that for any $x \in H$, $x^{-1} \in H$.
Let $x \in H$ and $x^m = x^n$ for some $n, m \in \mathbb{N}$ with $n \neq m$. If $n < m$, then

$$x^n * e = x^n = x^m = x^n * x^{m-n}$$

and $x^{m-n} = e$ by Cancellation.
If $m < n$, then

$$x^m * e = x^m = x^n = x^m * x^{n-m}$$

and $x^{n-m} = e$ by Cancellation. It follows that $x^t = e$ for some $t \in \mathbb{N}$.
If $t = 1$, then $x = e$ and $x^{-1} = e$. If $t > 1$, then $x * x^{-1} = e = x * x^{t-1}$ and $x^{-1} = x^{t-1} \in H$, so that $H \leq G$.

2. (a) Since $e \in L$ and $e \in K$, it follows that $e \in L \cap K$ and $L \cap K \neq \emptyset$.
Suppose $x, y \in L \cap K$. Since $L \leq G$, it follows by One-step test that $x * y^{-1} \in L$. Similarly, $x * y^{-1} \in K$ and so

$$x * y^{-1} \in L \cap K.$$

By One-step test, $L \cap K \leq G$.

(b) Since $K \leq L$, it follows that $K \neq \emptyset$. For any $x, y \in K$, $x * y^{-1} \in K$ as $K \leq L$. View x, y as elements of G. Since $L \leq G$, so $x * y^{-1} \in K$ and $K \leq G$ by One-step test.

(c) If $x * H = H$, then $x * H \leq G$. Conversely, suppose $x * H \leq G$. Then $e = e_G \in x * H$ and $e \in H$, so that $x * H \cap H \neq \emptyset$ and $x * H = H$.

3. (a) If $X, Y \in \mathrm{GL}_2(\mathbb{R})$, then X and Y are invertible, so is XY. Thus $\mathrm{GL}_2(\mathbb{R})$ is closed under multiplication.
If $X, Y, Z \in \mathrm{GL}_2(\mathbb{R})$, then $X(YZ) = (XY)Z$.
I_2 is the identity element in $\mathrm{GL}_2(\mathbb{R})$.
If $X \in \mathrm{GL}_2(\mathbb{R})$, then its inverse matrix X^{-1} is the inverse in $\mathrm{GL}_2(\mathbb{R})$. So $\mathrm{GL}_2(\mathbb{R})$ is a group.

(b) Since $\det(XY) = \det(X)\det(Y)$, it follows that det is a group homomorphism.

(c) Since $\det(I_2) = 1 = \det(-I_2)$, it follows that det is not one-to-one, so that it is not an isomorphism.

(d) Since $I_2 \in \mathrm{SL}_2(\mathbb{R})$, it follows that $\mathrm{SL}_2(\mathbb{R})$ is non-empty.
For $X, Y \in \mathrm{SL}_2(\mathbb{R})$, $\det(X) = \det(Y) = 1$ and so $\det(Y^{-1}) = \det(Y)^{-1} = 1$. Thus

$$\det(XY^{-1}) = \det(X)\det(Y^{-1}) = 1 \iff XY^{-1} \in \mathrm{SL}_2(\mathbb{R}).$$

By One-step test, $\mathrm{SL}_2(\mathbb{R}) \leq \mathrm{GL}_2(\mathbb{R})$.