1. Let $n = |A| = |B|$. Suppose f is one-to-one and $A = \{a_1, a_2, \ldots, a_n\}$. Then $f(a_i) \neq f(a_j)$ for all $i \neq j$, so $\mathcal{R}an(f) = \{f(a_1), f(a_2), \ldots, f(a_n)\}$ contains exactly n distinct elements, so that $|\mathcal{R}an(f)| = n$. But $|B| = n$ and $\mathcal{R}an(f) \subseteq B$, so $B = \mathcal{R}an(f)$ and f is onto.

Conversely, suppose f is onto and $B = \{b_1, b_2, \ldots, b_n\}$. Then $\exists\ x_i \in A\ f(x_i) = b_i$. Since $b_i \neq b_j$ for $i \neq j$, it follows that $x_i \neq x_j$, so that $A' := \{x_1, x_2, \ldots, x_n\}$ contains exactly n elements. But $|A| = n$ and $A' \subseteq A$, so $A = A'$ and f is one-to-one.

2. (a) Suppose $a, a' \in A$, Then

$$\begin{aligned} f(a) = f(a') \quad &\iff \quad \frac{2a}{a-1} = \frac{2a'}{a'-1} \\ &\iff \quad \frac{2(a-1)+2}{a-1} = \frac{2(a'-1)+2}{a'-1} \\ &\iff \quad 2 + \frac{2}{a-1} = 2 + \frac{2}{a'-1} \\ &\iff \quad \frac{2}{a-1} = \frac{2}{a'-1} \\ &\iff \quad a'-1 = a-1 \\ &\iff \quad a' = a, \end{aligned}$$

so f is one-one.

Now we show that f is onto.

For any $b \in B$, let $a = \frac{b}{b-2}$. Since $b \neq 2$, it follows that $a \in \mathbb{R}$. If $a \notin A$, then $a = 1$, i.e. $b = b - 2$ or $0 = -2$, which is impossible. Thus $a \in A$ and

$$f(a) = \frac{2\frac{b}{b-2}}{\frac{b}{b-2} - 1} = \frac{2b}{b-(b-2)} = b,$$

so f is onto.

(b) As shown above $f^{-1} : B \to A$ is given by $f^{-1}(x) = \frac{x}{x-2}$.

(c) Since $f^{-1} \circ f = 1_A$, it follows that $f \circ f^{-1} \circ f = f \circ 1_A = f$.

3. Suppose $g \circ f$ is onto and g is one-to-one. Then f is onto $\iff$ $(\forall b \in B)(\exists\ a \in A)(f(a) = b)$.

Suppose $b \in B$ and let $c = g(b)$. Since $g \circ f$ is onto, $g \circ f(a) = c$ for some $a \in A$. Thus $g(f(a)) = c = g(b)$. But g is one-to-one, so $b = f(a)$ and f is onto.

4. (a) Suppose first that F is not 1-1. We must show that f is not 1-1.

There exist $P, Q \subseteq A, P \neq Q$ with $F(P) = F(Q)$. But $P \neq Q \implies (\exists x \in P \backslash Q)) \vee (\exists y \in Q \backslash P)$. Suppose without loss of generality that $x \in P \setminus Q$. Then since $F(P) = F(Q), \exists y \in Q : f(y) =$

$f(x) \in F(P) = F(Q)$. So $(\exists y \in Q)(f(y) = f(x))$ but $x \neq y$ since $x \notin Q$ and $y \in Q$. Hence f is not 1-1.

Conversely if f is not 1-1. Then $\exists x, y : x \neq y$ such that $f(x) = f(y)$. Then $F(\{x\}) = F(\{y\}) = \{f(x)\} = \{f(y)\}$ but $\{x\} \neq \{y\}$ so F not 1-1.

(b) Suppose first that F is onto. We must show that f is onto. Let $b \in B$, so that $Y := \{b\} \in \mathcal{P}(B)$. Since F is onto, $F(X) = Y$ for some $X \in \mathcal{P}(A)$. Since $F(\emptyset) = \emptyset \neq Y$, it follows that $X \neq \emptyset$, so that $(\exists\, a \in X)(f(a) \in Y) \iff (\exists\, a \in X)(f(a) = b)$. Thus f is onto.

Conversely, suppose f is onto and let $Y \in \mathcal{P}(B)$. If $X = f^{-1}(Y) := \{x \in A : f(x) \in Y\}$, then $X \in \mathcal{P}(A)$ and $F(X) \subseteq Y$. If $y \in Y$, then $f(x) = y$ for some $x \in A$, since f is onto, so that $x \in X$ and $F(X) = Y$. Thus F is onto.

5. Note that $A = \{1 - \frac{1}{2}, 1 - \frac{1}{2^2}, 1 - \frac{1}{2^3}, \ldots\} = \{1 - \frac{1}{2} < 1 - \frac{1}{2^2} < 1 - \frac{1}{2^3} < \ldots\}$.

Define $f : A \to \mathbb{N}$ by $f(1 - \frac{1}{2^n}) = n$. Then f is a function.

f **is onto**. For any $b \in \mathbb{N}$, set $a = 1 - \frac{1}{2^b}$. Then $a \in A$ and $f(a) = b$.

f **is strictly order preserving**. Let $a = 1 - \frac{1}{2^n}$ and $a' = 1 - \frac{1}{2^{n'}}$ be two elements of A for some $n, n' \in \mathbb{N}$. Then

$$\begin{aligned} a \leq a' &\iff 1 - \frac{1}{2^n} \leq 1 - \frac{1}{2^{n'}} \\ &\iff -\frac{1}{2^n} \leq -\frac{1}{2^{n'}} \\ &\iff \frac{1}{2^{n'}} \leq \frac{1}{2^n} \\ &\iff 2^n \leq 2^{n'} \\ &\iff n \leq n' \\ &\iff f(a) \leq f(a'). \end{aligned}$$

Thus f is an order isomorphism and so $A \simeq \mathbb{N}$.