1. For $n \in \mathbb{N}$, let P_n be the statement

$$\text{“}5 \mid (n^5 - n)\text{''}.$$

Base case: When $n = 1$, $1^5 - 1 = 0 = 5 \cdot 0$. Thus P_1 is true.

Inductive step: For $k \in \mathbb{N}$ suppose P_k is true, that is, $5 \mid (n^5 - n)$, which is equivalent to $5c = k^5 - k$ for some $c \in \mathbb{N} \iff k^5 = 5c + k$ for some $c \in \mathbb{N}$.

Want to show that P_{k+1} is true, i.e. $5 \mid ((k+1)^5 - (k+1))$. Now

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1) \\
&= 5c + k + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= 5(c + k^4 + 2k^3 + 2k^2 + k).
\end{aligned}$$

Thus P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$.

2. $x_1 = 1, x_2 = 4, x_3 = 2x_2 - x_1 + 2 = 8 - 1 + 2 = 9, x_4 = 2x_3 - x_2 + 2 = 18 - 4 + 2 = 16$ and $x_5 = 2x_4 - x_3 + 2 = 32 - 9 + 2 = 25$. We conjecture that $x_n = n^2$.

For $n \in \mathbb{N}$, let P_n be the statement $x_n = n^2$.

Base case: $x_1 = 1 = 1^2$ and $x_2 = 4 = 2^2$, so that $x_n = n^2$ for $n = 1$ and $n = 2$.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_i is true for all $1 \leq i \leq k$, that is, $x_i = i^2$ for all $1 \leq i \leq k$.

Want to show that P_{k+1} is true, that is, $x_{k+1} = (k+1)^2$. Since $k + 1 \geq 3$,

$$\begin{aligned}
x_{k+1} &= 2x_k - x_{k-1} + 2 \\
&= 2k^2 - (k-1)^2 + 2 \qquad \text{as} \qquad x_i = i^2 \\
&= 2k^2 - (k^2 - 2k + 1) + 2 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2,
\end{aligned}$$

that is, $x_{k+1} = (k+1)^2$. Thus P_{k+1} is true and by complete induction, P_n is true for all $n \in \mathbb{N}$, that is $x_n = n^2$.

3. For $n \in \mathbb{N}$ with $n \geq 2$, let P_n be the statement that

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^C_U = (A_1)^C_U \cup (A_2)^C_U \cup \ldots \cup (A_n)^C_U.$$

Base case: When $n = 2$, P_2 is the statement $(A_1 \cap A_2)_U^C = (A_1)_U^C \cup (A_2)_U^C$. Now

$$\begin{aligned} x \in (A_1 \cap A_2)_U^C &\iff (x \in U) \wedge (x \notin (A_1 \cap A_2)) \\ &\iff (x \in U) \wedge (x \notin A_1 \vee x \notin A_2) \\ &\iff (x \in U \wedge x \notin A_1) \vee (x \in U \wedge x \notin A_2) \\ &\iff x \in (A_1)_U^C \vee x \in (A_2)_U^C \\ &\iff x \in (A_1)_U^C \cup (A_2)_U^C, \end{aligned}$$

that is, $(A_1 \cap A_2)_U^C = (A_1)_U^C \cup (A_2)_U^C$ and P_2 is true.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_k is true, that is,

$$(A_1 \cap A_2 \cap \ldots \cap A_k)_U^C = (A_1)_U^C \cup (A_2)_U^C \cup \ldots \cup (A_k)_U^C.$$

If $T = A_1 \cap A_2 \cap \ldots \cap A_k$, then

$$\begin{aligned} (A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1})_U^C &= (T \cap A_{k+1})_U^C \\ &= T_U^C \cup (A_{k+1})_U^C \\ &= (A_1 \cap \ldots \cap A_k)_U^C \cup (A_{k+1})_U^C \\ &= (A_1)_U^C \cup \ldots \cup (A_k)_U^C \cup (A_{k+1})_U^C, \end{aligned}$$

that is, P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$.

4. For $n \in \mathbb{N}$, let P_n be the statement $(1+x)^n \geq 1 + nx$.

Base case: P_1 is true, because $(1+x)^1 = 1 + 1 \cdot x \geq 1 + 1 \cdot x$.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_k is true, that is, $(1+x)^k \geq 1 + kx$. Want to show that $(1+x)^{k+1} \geq (1 + (k+1)x)$. Now

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \\ &\geq (1+kx)(1+x) \qquad \text{as} \qquad (1+x)^k \geq 1+kx \qquad \text{and} \qquad (1+x) > 0 \\ &= 1 + kx + x + kx^2 \\ &\geq 1 + kx + x \qquad \text{as} \qquad kx^2 \geq 0 \\ &= 1 + (k+1)x, \end{aligned}$$

that is, $(1+x)^k \geq 1 + kx$. Thus P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$, that is, $(1+x)^n > 1 + nx$.