1. (a) We have the truth table

A	B	$A \iff B$	$A \implies B$	$B \implies A$	$(A \implies B) \wedge (B \implies A)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

The columns for $A \iff B$ and $(A \implies B) \wedge (B \implies A)$ are identical. Therefore $A \iff B$ and $(A \implies B) \wedge (B \implies A)$ are logically equivalent, that is, $(A \iff B) \iff ((A \implies B) \wedge (B \implies A))$ is a tautology.

(b) We have the truth table

A	B	C	$A \vee B$	$B \vee C$	$A \implies (B \vee C)$	$\iff$	$(A \vee B) \implies C$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	F	T	T	T

Since the column for $\iff$ contains both "T" and "F", $(A \iff B) \iff ((A \implies B) \wedge (B \implies A))$ is neither a tautology nor a contradiction.

2. (a) Suppose n is an integer. Let $F(n) :=$ "$(\exists q \in \mathbb{Z})(n = 4q + 1)''$, $G(n) :=$ "$(\exists q \in \mathbb{Z})(n = 4q + 2)''$ and $O(n) :=$ "n is odd". Then

$$A(n) = ((F(n) \vee G(n)) \implies O(n^2)).$$

(b) $\sim A(n) \iff ((F(n) \vee G(n)) \wedge E(n^2))$, so the negation of $A(n)$ is "there exists $q \in \mathbb{Z}$ such that $n = 4q + 1$ or $n = 4q + 3$ but n^2 is even.

(c) The converse of $A(n)$ is $(O(n^2) \implies (F(n) \vee G(n)))$, namely, if n^2 is odd then $n = 4q + 1$ or $4q + 3$ for some $q \in \mathbb{Z}$.

(d) The contrapositive of $A(n)$ is $(\sim O(n^2) \implies \sim((F(n) \vee G(n))) \iff (E(n^2) \implies (\forall q \in \mathbb{Z})(n \neq 4q + 1) \wedge (\forall q \in \mathbb{Z})(n \neq 4q + 3))$, that is, if n^2 is even, then $n \neq 4q + 1$ and $n \neq 4q + 3$ for any $q \in \mathbb{Z}$.

(e) Suppose $n = 4q + 1$ or $4q + 3$ for some $q \in \mathbb{Z}$.
If $n = 4q+1$, then $n^2 = (4q+1)^2 = 16q^2 + 8q + 1 = 2(8q^2 + 4q) + 1$ and so n^2 is odd as $8q^2 + 4q$ is an integer.

If $n = 4q+3$, then $n^2 = (4q+3)^2 = 16q^2 + 24q + 9 = 2(8q^2 + 12q + 4) + 1$ and n^2 is also odd as $8q^2 + 12q + 4$ is an integer.

It follows that n^2 is odd.

(f) The converse of $A(n)$ is that if n^2 is odd then $n = 4q+1$ or $4q+3$ for some $q \in \mathbb{Z}$.

Suppose, for a contradiction that n^2 is odd but $n \neq 4q+1$ or $n \neq 4q+3$ for any $q \in \mathbb{Z}$. Then $n = 4\ell$ or $4\ell + 2$ for some integer $\ell \in \mathbb{Z}$, so that n is even. Thus n^2 is even and so n^2 is both even and odd, which is a contradiction. It follows that if n^2 is odd then $n = 4q+1$ or $4q+3$ for some $q \in \mathbb{Z}$.

(g) The contrapositive of $A(n)$ is true, since it is equivalent to $A(n)$ and by (b) $A(n)$ is true for all $n \in \mathbb{Z}$.

3. (a) $\iff (\forall x, y, z \in \mathbb{Z})(O(x+z) \implies (O(x+y) \wedge O(y+z)))$. So $\sim(a) \iff (\exists x, y, z \in \mathbb{Z})(O(x+z) \wedge (E(x+y) \vee E(y+z)))$.

(1) Take $x = 1$, $y = 2$ and $z = 2$.

(2) $x + z = 3$ is odd and $y + z = 4$ is even.

Thus $x = 1$, $y = 2$ and $z = 2$ is a counterexample to the statement (a).

(b) $\iff (\exists n \in \mathbb{Z})E(n^3 + n + 1)$. So $\sim(a) \iff (\forall n \in \mathbb{Z})O(n^3 + n + 1)$, namely, $n^3 + n + 1$ is odd for any $n \in \mathbb{Z}$.

Suppose n is an integer. If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$ and so

$$n^3 + n + 1 = 8k^3 + 2k + 1 = 2(4k^3 + k) + 1.$$

Since $4k^3 + k \in \mathbb{Z}$, it follows that $n^3 + n + 1$ is odd.

If n is odd, then $n = 2k+1$ for some $k \in \mathbb{Z}$ and so

$$n^3 + n + 1 = 8k^3 + 12k^2 + 6k + 1 + 2k + 1 + 1 = 2(4k^3 + 6k^2 + 4k + 1) + 1.$$

Since $4k^3 + 6k^2 + 4k + 1 \in \mathbb{Z}$, it follows that $n^3 + n + 1$ is odd. Thus $n^3 + n + 1$ is odd for any $n \in \mathbb{Z}$.

(c) $\iff (\forall a, c \in \mathbb{Z})(P(a) \wedge P(c) \implies (\exists b \in \mathbb{Z})(P(b) \wedge S(a,b,c)))$, where $P(x) :=$ "$x > 0$" and $S(x,y,z) :=$ "$x + y = z$". So $\sim(c) \iff (\exists a, c \in \mathbb{Z})(P(a) \wedge P(c) \wedge (\forall b \in \mathbb{Z})(\sim P(b) \vee \sim S(a,b,c)))$.

(1) Take $a = 2$ and $c = 1$.

(2) $a > 0$ and $c > 0$. For any $b \in \mathbb{Z}$ either $a + b \neq c$ or $a + b = c$ but $b = c - a = 1 - 2 < 0$, that is, $(\forall b \in \mathbb{Z})(\sim P(b) \vee \sim S(a,b,c))$.

Thus $a = 2$ and $c = 1$ is a counterexample to the statement (c).