DEPARTMENT OF MATHEMATICS

MATHS 255 FS	Answers for Collaborative Tutorial 4	May 11, 2005

1. (a) Note that in $\mathbb{Z}_7$,

x	1	2	3	4	5	6
$3 \cdot_7 x$	3	6	2	5	1	4

Thus $3^{-1} = 5$ in $\mathbb{Z}_7$, and

$$
\begin{array}{rrrrrrr}
 & & & 5x^3 & & +6x & +3 \\ \cline{2-7}
3x^2+2 &) & x^5 & & +2x^2 & +x & +4 \\
 & & x^5 & +3x^3 & & & \\ \cline{3-7}
 & & & 4x^3 & +2x^2 & +x & +4 \\
 & & & 4x^3 & & +5x & \\ \cline{4-7}
 & & & & 2x^2 & -4x & +4 \\
 & & & & 2x^2 & & +6 \\ \cline{5-7}
 & & & & & 3x & +5
\end{array}
$$

So $q(x) = 5x^3 + 6x + 5$ and $r(x) = 3x + 5$.

(b) Use long division again

$$
\begin{array}{rrrrr}
 & & & x & +3 \\ \cline{2-5}
3x+5 &) & 3x^2 & & +2 \\
 & & 3x^2 & +5x & \\ \cline{2-5}
 & & & 2x & +2 \\
 & & & 2x & +1 \\ \cline{4-5}
 & & & & 1
\end{array}
$$

The algorithm gives us

$d(x)$	$u(x)$	$v(x)$	
$f(x)$	1	0	r_1
$g(x)$	0	1	r_2
$3x+5$	1	$-(5x^2+6x+3)$	$r_3 = r_1 - (5x^2+6x+3)r_2$
1	$(6x+4)$	$5x^4+x^3+6x^2+3$	$r_4 = r_2 - (x+3)r_3$
0	$*$	$*$	$r_5 = r_3 - (3x+5)r_4$

from which we see that $\gcd(f(x), g(x)) = 1$ and

$$1 = f(x)(6x+4) + g(x)(5x^4 + x^3 + 6x^2 + 3),$$

so that $u(x) = 6x + 4$ and $v(x) = 5x^4 + x^3 + 6x^2 + 3$.

2. (a) For all $a, b \in \mathbb{R}$

$$(1+a)(1+b) = 1 + a + b + ab = 1 + a * b.$$

If $*$ is not a binary operation on A, then $a * b \notin A$ for some $a, b \in A$, so that $a * b = -1$, since $a * b \in \mathbb{R}$ and $A = \mathbb{R} \backslash \{-1\}$. Thus $(1+a)(1+b) = 1 + a * b = 0$ and $a = -1$ or $b = -1$. Contradiction.

For $a, b, c \in A$, $((1+a)(1+b))(1+c) = 1 + (a*b)*c$ and $(1+a)((1+b)(1+c)) = 1 + a*(b*c)$. Since $((1+a)(1+b))(1+c) = (1+a)((1+b)(1+c))$ in $\mathbb{R}$, it follows that $(a*b)*c = a*(b*c)$.

(b) 0 is the identity of $(A, *)$, since $0 * a = 0 + a + 0a = a = a * 0$ for all $a \in A$.

For $a \in A$, if b is its inverse, then $a * b = 0 \iff (1+a)(1+b) = 1 \iff b = \frac{-a}{1+a}$. Note that $b = \frac{-a}{1+a} \neq -1$, since $(1+a)(1+b) = 1$. Check $a * \frac{-a}{1+a} = \frac{-a}{1+a} * a = 0$, so $\frac{-a}{1+a}$ is the inverse of a. Thus $(A, *)$ is a group.

Since $a * b = a + b + ab = b * a$ for any $a, b \in G$, it follows that $(A, *)$ is an abelian group.

3. (a) $e * e^{-1} = e = e * e \implies e^{-1} = e$ by Cancellation low.

(b) $x * x^{-1} = e = x * x^{n-1} \implies x^{-1} = x^{n-1}$ by Cancellation low.