1. (a) Write $\sim = \rho$. **Reflexive**: $(\forall\ X \in S)((x \in X) \implies (\exists x \in X)(x|x))$, that is, $X \sim X$.

Not antisymmetric: Take $X = \{2, 4\}$ and $Y = \{2\}$. Then $X \sim Y$ and $Y \sim X$, but $X \neq Y$.

Not symmetric: Take $X = \{2\}$ and $Y = \{2, 3\}$. Then $X \sim Y$, but $Y \not\sim X$ since $3 \in Y$ but $2 \not| 3$.

Transitive: $(X \sim Y) \wedge (Y \sim Z) \iff (\forall\ x \in X)(\exists\ y \in Y)(y|x) \wedge (\forall\ y \in Y)(\exists\ z \in Z)(z|y)$. If $x \in X$ and $y|x$ for some $y \in Y$, then $z|y$ for some $z \in Z$, so $z|x$. Thus $(\forall\ x \in X)(\exists\ z \in Z)(z|x)$, and so $X \sim Z$.

(b) Write $\sim = \rho$. **Not reflexive**: Take $x = 3 \in B$. Then $3 \sim 3 \iff 3t = 3$ for some $t \in B$, so that $t = 1 \notin B$. Contradiction.

Antisymmetric: If $x, y \in B$ with $x \sim y$ and $y \sim x$, then $xt = y$ and $ys = x$ for some $t, s \in B$. So $x = tsx$ and $ts = 1$ for some $s, t \in B$, which is impossible. Thus $(x \sim y) \wedge (y \sim x)$ is always false and hence $(x \sim y) \wedge (y \sim x) \implies (x = x)$.

Not symmetric: Take $x = 3$ and $y = 9$. Then $3 \sim 9$ as $3 \cdot 3 = 9$ and $3 \in B$. But $9s \neq 3$ for any $s \in B$, so $9 \not\sim 3$.

Transitive: $(x \sim y) \wedge (y \sim z) \iff (\exists\ t \in B)(xt = y) \wedge (\exists\ s \in B)(ys = z) \implies (\exists\ ts \in B)(x(ts) = y) \iff x \sim z$.

(c) Note that $\rho = \{(a, b) \in C \times C : a + b = 6\} = \{(2, 4), (4, 2)\}$. Thus ρ is **not reflexive**, **not antisymmetric**, **symmetric** and **not transitive**.

(d) Note that $\rho = \{(a, b) \in D \times D : a + 2b = 6\} = \emptyset$. Thus ρ is **not reflexive**, **antisymmetric**, **symmetric** and **transitive**.

2. (a) For all $x \in S$, $(x, x) \in \rho$, so ρ is reflexive.

For all $x, y \in S$, $(x, y) \in \rho \implies (y, x) \in \rho$, so ρ is symmetric.

For all $x, y, z \in S$, $((x, y) \in \rho \wedge (y, z) \in \rho) \implies (x, z) \in \rho$, so ρ is transitive.

$[1] = T_1 = \{1, 2, 3\}$, $[4] = T_4 = \{4, 5\}$ and $[6] = T_6 = \{6\}$.

(b) $S_i \neq \emptyset$ for each i, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S = S_1 \cup S_2 \cup S_3$. So $\{S_1, S_2, S_3\}$ is a partition.

$$\rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 2), (2, 1), (3, 4), (4, 3), (3, 6), (6, 3)(4, 6), (6, 4)\}.$$

3. (a) **Reflexive**: $(\forall\ x \in A)(3|x + 2x = 3x)$ so $x \sim x$.

Symmetric: $x \sim y \iff 3|(x + 2y) \iff (\exists\ a \in \mathbb{Z})(x + 2y = 3a)$. Thus $y + 2x = 3y + 3x - (x + 2y) = 3(y + x - a)$ and $3|(y + 2x)$, so $y \sim x$.

Transitive: $(x \sim y) \wedge (y \sim z) \iff (\exists\ a \in \mathbb{Z})(x + 2y = 3a) \wedge (\exists\ b \in \mathbb{Z})(y + 2z = 3b)$. Thus $x + 2z = (x + 2y) + (y + 2z) - 3y = 3(a + b - y)$, so $3|(x + 2z)$ and $x \sim z$.

(b) $x \in [0] \iff 3|(x + 2 \cdot 0) \iff x = 3t$ for some $t \in \mathbb{Z}$, $x \in [1] \iff 3|(x + 2) \iff x = 3t - 2 = 3(t - 1) + 1$ for some $t \in \mathbb{Z}$ and $x \in [2] \iff 3|(x + 4) \iff x = 3t - 4 = 3(4 - 2) + 2$ for some $t \in \mathbb{Z}$. Thus $\mathbb{Z} = [0] \cup [1] \cup [2]$ and so $[0], [1], [2]$ are all the distinct equivalence classes.