1.

$$\begin{aligned} x \in (A \cap B)^C_U &\iff (x \in U) \wedge (x \notin (A \cap B)) \\ &\iff (x \in U) \wedge \sim(x \in A \wedge x \in B) \\ &\iff (x \in U) \wedge ((x \notin A) \vee (x \notin B)) \\ &\iff (x \in U \wedge x \notin A) \vee (x \in U \wedge x \notin B) \\ &\iff x \in A^C_U \cup B^C_U. \end{aligned}$$

Thus

$$(A \cap B)^C_U = A^C_U \cup B^C_U.$$

2.

$$\begin{aligned} (a,b) \in A \times (B \setminus C) &\iff (a \in A) \wedge (b \in B \setminus C) \\ &\iff (a \in A) \wedge (b \in B \wedge b \notin C) \\ &\iff ((a \in A) \wedge (b \in B)) \wedge ((a \in A) \wedge (b \notin C)) \\ &\iff ((a,b) \in A \times B) \wedge ((a,b) \notin A \times C) \\ &\iff (a,b) \in (A \times B) \setminus (A \times C) \end{aligned}$$

Thus

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$

3. Use proof by contraposition.

Suppose $(A \times B) \cap (B \times A) \neq \emptyset$. Let (a, b) be an element of $(A \times B) \cap (B \times A)$. Then

$$((a \in A) \wedge (b \in B)) \wedge ((a \in B) \wedge (b \in A)),$$

and in particular, $a \in A \cap B$ and hence $A \cap B \neq \emptyset$.

Conversely suppose $A \cap B \neq \emptyset$ and let $a \in A \cap B$. Then $(a, a) \in (A \times B) \cap (B \times A)$, so that $(A \times B) \cap (B \times A) \neq \emptyset$.

It follows that

$$(A \times B) \cap (B \times A) = \emptyset \iff A \cap B = \emptyset.$$

4. Suppose $A \cap B = A \cap C$ and $A \cup B = A \cup C$. Suppose $a \in B$.

If $a \in A$, then $a \in A \cap B = A \cap C$ and so $a \in C$, since $A \cap C \subseteq C$.

If $a \notin A$, then $a \in A \cup B$ but $a \notin A$. Since $A \cup B = A \cup C$, it follows that $a \in A \cup C$ but $a \notin A$, so that $a \in C$.

It follows that $B \subseteq C$. Similarly, $C \subseteq B$ and so $B = C$.