NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. (**10 marks**) Let $X = \{a, b, c, d\}$ be a set of four elements and define $*$ by $x * y = y$ for all $x, y \in X$.

(a) (**3 marks**) Show that $*$ is a binary operation on X and construct its Cayley Table.

(b) (**3 marks**) Show that $*$ is associative but non-commutative.

(c) (**2 marks**) Determine whether or not $(X, *)$ has an identity.

(d) (**2 marks**) Determine whether or not $(X, *)$ is a group.

2. (**20 marks**) Let $G = D_4$ be the group of symmetries of the square in the plane. The Cayley table is shown below

$*$	e	φ	π	ϕ	h	v	d	d'
e	e	φ	π	ϕ	h	v	d	d'
φ	φ	π	ϕ	e	d'	d	h	v
π	π	ϕ	e	φ	v	h	d'	d
ϕ	ϕ	e	φ	π	d	d'	v	h
h	h	d	v	d'	e	π	φ	ϕ
v	v	d'	h	d	π	e	ϕ	φ
d	d	v	d'	h	ϕ	φ	e	π
d'	d'	h	d	v	φ	ϕ	π	e

(a) (**3 marks**) Show that D_4 is non-Abelian.

(b) (**4 marks**) List the elements which are their own inverse.

(c) (**4 marks**) Find two distinct subgroups of order 2.

(d) (**5 marks**) Find two distinct subgroups of order 4.

(e) (**4 marks**) Explain why there are no subgroups of G containing five elements.

3. (**18 marks**)

(a) Let A be a non-empty bounded subset of $\mathbb{R}$, and $B = \{y \in \mathbb{R} : y = -x,\ \exists\ x \in A\}$.

(i) (**3 marks**) Show that B is bounded.

(ii) (**5 marks**) If L is the least upper bound of B, then show that $-L$ is the greatest lower bound of B.

(iii) (**5 marks**) Suppose M is a lower bound of B, and suppose $(\forall\ \epsilon > 0)(\exists\ y \in B)(M \le y \le M + \epsilon)$. Show that M is the greatest lower bound of B.

(b) (**5 marks**) Let X be a non-empty subset of $\mathbb{R}$ which is bounded and $s \in \mathbb{R}$ with $s > 0$. Let $Y := \{y : y = sx,\ \exists\ x \in X\}$. Show that

$$\text{glb } Y = s \text{ glb } X.$$

4. (**10 marks**) Suppose F is a field with finite number of elemenets. Write $nx = x + x + \ldots + x$ (n terms).

(a) (**4 marks**) Show that there is a positive integer m such that

$$m1_F = 0_F,$$

(b) (**3 marks**) Suppose p is the smallest positive integer such that $p1_F = 0_F$. Show that p is prime. (The number p is called the *characteristic* of the field F).

(c) (**3 marks**) Show that for every $x \in F$ $px = 0_F$, where p is given by (b) above.

5. (**7 marks**) Prove from the first principles that the sequence $\{\frac{2n+3}{5n-2}\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$.

6. (**15 marks**) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and define the sequence $\{c_n\}_{n=1}^{\infty}$ by

$$c_n = \begin{cases} a_n & \text{if } b_n \leq a_n, \\ b_n & \text{if } a_n < b_n. \end{cases}$$

Suppose that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$$

for some real number $a \in \mathbb{R}$. Prove from the first principles that $\lim_{n\to\infty} c_n = a$.