NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. (**15 marks**)

(a) Find all solutions to the following Diophantine equations:

(i) (**4 marks**) $946x + 374y = 18$.

(ii) (**4 marks**) $976x + 3742y = 44$.

(b) (**7 marks**) Find all solutions to the Diophantine equation $976x + 374y = 22$ with $0 \leq x \leq 40$

2. (**20 marks**)

(a) (**5 marks**) Find all integers $x \in \mathbb{Z}$ such that

$$2x^2 - 3x - 4 \equiv 0 \pmod 5.$$

(b) (**7 marks**) Find all integers $x \in \mathbb{Z}$ such that

$$189x \equiv 28 \pmod{56}.$$

(c) (**8 marks**) Find the smallest positive solution $x \in \mathbb{Z}$ such that

$$946x \equiv 26 \pmod{2316}.$$

3. (**8 marks**) Use congruences to show that for any natural number $n \in \mathbb{N}$, the number $21(15n + 27)(n + 28)$ is divisible by 14.

4. (**17 marks**)

(a) (**5 marks**) Let $a(x) = x^3 - 2x - 1$ and $b(x) = x^3 + 5x^2 + 2x - 2$ be polynomials of $\mathbb{R}[x]$. Use the Euclidean Algorithm for $\mathbb{R}[x]$ to find the greatest common monic divisor in $\mathbb{R}[x]$.

(b) Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{Z}_5[x]$ defined by

$$f(x) = x^4 + 2x^3 + 4x + 1, \quad g(x) = 3x^3 + x^2 + x + 2.$$

Here for simplicity, we denote $\bar{a}$ by a for $\bar{a} \in \mathbb{Z}_5$.

(i) (**6 marks**) Find quotient $q(x)$ and remainder $r(x)$ when $f(x)$ is divided by $g(x)$.

(ii) (**6 marks**) Find the greatest common monic divisor of $f(x)$ and $g(x)$ and find polynomials $u(x)$ and $v(x)$ such that

$$d(x) = f(x)u(x) + g(x)v(x).$$

5. (**8 marks**) Let $(G, *)$ be a group with identity e, and $a, b, c \in G$.

(a) If $a * b = c * b$, then $a = c$.

(b) If $a * b = e$, then $b * a = e$.

6. (**12 marks**) Let $A = \{x \in \mathbb{R} : x \neq 0\}$ be the set of all non-zero real numbers and $T = \mathbb{R} \setminus \mathbb{Q}$. For any $x, y \in \mathbb{R}$ define $x * y$ by

$$x * y = 3xy,$$

where xy is the ordinary multiplication of x and y in $\mathbb{R}$.

(a) (**9 marks**) Show that $(A, *)$ is an abelian group.

(b) (**3 marks**) Show that $*$ is not a binary operation on T.