NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. (**18 marks**) Let $A = \{x \in \mathbb{Z} : -9 \leq x \leq 9\}$. Let $f : A \to A$ be defined as follows: For all $x \in A$, $f(x)$ is the remainder when x is divided by 5. [You are not asked to prove that f is a function.]

(a) (i) Find $f(7)$ and $f(-7)$.
(ii) Determine whether or not f is one-to-one.
(iii) Determine whether or not f is onto.

(b) Let $g : \mathcal{P}(A) \to \mathcal{P}(A)$ be defined as follows: For all $X \in \mathcal{P}(A), g(X) = \{a \in A : f(a) \in X\}$. [You are not asked to prove that g is a function.]
(i) What is $g(\{-1, 0, 1\})$?
(ii) Determine whether or not g is one-to-one.
(iii) Determine whether or not g is onto.

(c) A relation is defined on A as follows: For all $a, b \in A$, $a \sim b$ if and only if $f(a) = f(b)$.
(i) Show that $\sim$ is an equivalence relation.
(ii) List all elements of the set $S = \{a \in A : a \sim 7\}$.
(iii) Write down all of the equivalence classes under the relation $\sim$.

2. (**8 marks**) Let $A = \{x \in \mathbb{R} : x \neq 3\}$, $B = \{x \in \mathbb{R} : x \neq 5\}$ and define $f : A \to B$ by $f(x) = \frac{5x}{x-3}$.

(a) (**6 marks**) Show that f is one-to-one and onto.
(b) (**2 marks**) Determine the inverse f^{-1} of f.

3. (**15 marks**) For $x \in \mathbb{Z}$, define a function f by

$$\begin{aligned} f(x) &= 2x + 2 \quad \text{if} \quad x \geq 0, \\ &= -2x - 1 \quad \text{if} \quad x < 0. \end{aligned}$$

(a) (**6 marks**) Show that f is a bijection from $\mathbb{Z}$ to $\mathbb{N}$.
(b) (**3 marks**) Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is one-to-one but not onto
(c) (**4 marks**) Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is onto but not one-to-one
(d) (**2 marks**) Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is neither one-to-one nor onto.

4. (**10 marks**)

(a) Let $A = \{-\frac{1}{n} : n \in \mathbb{N}\}$. Show that $(A, \leq) \simeq (\mathbb{N}, \leq)$ as posets.
(b) Let $B = \{-\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$. Show that $(B, \leq) \not\simeq (\mathbb{Z} \setminus \{0\}, \leq)$ as posets.

5. (**20 marks**)

(a) Let $a, b \in \mathbb{Z}$ not both zero, and $d = \gcd(a, b)$. If $a = da_1$ and $b = db_1$ for some $a_1, b_1 \in \mathbb{Z}$, then show that $\gcd(a_1, b_1) = 1$.

(b) Let $p \in \mathbb{N}$ be a prime number and $c \in \mathbb{Z}$. Show that either $p|c$ or $\gcd(c, p) = 1$.

(c) Let $w \in \mathbb{Z}$ be a odd number. Show that w and $w + 2$ are relatively prime.

(d) Let a, b be natural numbers. Show that there are infinitely many pairs $s, t \in \mathbb{Z}$ such that $\gcd(a, b) = as + bt$.

6. (**9 marks**) Use the modified version of Euclid's Algorithm to find $\gcd(a, b)$ and integers x and y with $\gcd(a, b) = ax + by$ for the following pairs of integers.

(a) 51 and 288.

(b) 357 and 629.

(c) 180 and 252.