NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. (**6 marks**) Translate the following statements into symbols and disprove each of them.

(a) For any integers $a, b, c \in \mathbb{Z}$, if ab, ac and bc are even, then a, b, and c are even.

(b) There are integers $x, y \in \mathbb{Z}$ such that xy is even but both x and y are odd.

2. (**7 marks**) Use induction to prove that $7 \mid (4^{2n} - 2^n)$ for any natural number n.

3. (**7 marks**) Use induction to prove that for every integer $n \geq 4$, $n! > 2^n$.

4. (**8 marks**) Let x be an integer. Show by induction that for all $n \in \mathbb{N}$, x^n is even if and only if x is even.

5. (**8 marks**) Let $x_1, x_2, x_3, \ldots$ be a sequence of integers defined recursively by

$$x_1 = 3, \quad x_2 = 18, \quad \text{and} \quad x_n = 6x_{n-1} - 9x_{n-2}$$

for integers $n \geq 3$. Conjecture a formula for x_n and prove it using mathematical induction.

6. (**12 marks**) Indicate whether each of the following relations on the given set is reflexive, symmetric, antisymmetric, or transitive. Explain your answers.

(a) $A = \{x \in \mathbb{Z} : x < 0\}$ and $x \rho y$ iff $2x + y = 0$.

(b) $B = \{x \in \mathbb{Q} : x \geq 0\}$ and $x \rho y$ iff $x + y = 1$.

(c) $C = \{a, b, c\}$ and $\rho = \{(a, b), (b, a), (a, c), (b, c), (a, a)\}$.

(d) $D = \mathbb{Q}$ and $x \rho y$ iff $|x - y| < 2$.

7. (**10 marks**) Let A be a set and R a relation on A.

(a) (**3 marks**) Show that there is an equivalence relation Q on A such that $R \subseteq Q$.

(b) (**4 marks**) Let Ω be the set consisting of all equivalence relations containing R and set

$$S := \bigcap_{Q \in \Omega} Q.$$

Show that S is an equivalence relation on A and $R \subseteq S$.

(c) (**3 marks**) Show that if X is an equivalence relation containing S, then $S \subseteq X$, that is, S is the "smallest" equivalence relation containing S.

8. (**10 marks**) Let $\sim$ be the relation defined on the set of integers $\mathbb{Z}$ by $x \sim y$ if $8 \mid (3x + 5y)$ for $x, y \in \mathbb{Z}$.

(a) (**7 marks**) Show that $\sim$ is an equivalence relation.

(b) (**3 marks**) Find the equivalence class $[0]$ containing 0.

9. (**12 marks**) Let $S = \{2, 4, 5, 6, 8, 10, 15, 18, 20\}$ and let ρ be the relation on S defined by $a \ \rho \ b$ if and only if $a \mid b$. Then (S, ρ) is a poset. [You are not asked to prove this.]

(a) Draw a lattice diagram of (S, ρ).

(b) Find all maximal and all minimal elements of S.

(c) Find a subset of S which has no upper bound and no lower bound.

(d) Find the greatest lower bound for $\{4, 6, 10\}$.

(e) Determine whether or not the subset $\{2, 4, 20\}$ of S is totally ordered. Explain your answer.