**NB:** Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

PLEASE SHOW ALL WORKING.

**1.** (**10 marks**) Which of the following sentences are statements, which are predicates, and which are neither? Translate all the statements and predicates into symbols.

(a) Every even integer is a multiple of 4.

(b) If $n$ is a prime number then $n^2$ is not even.

(c) $x^2$ is positive.

(d) Find an even number.

(e) For any integer $n$ there is an even number $m$ such that $m + n = -n$.

**2.** (**12 marks**) Let $A$, $B$ and $R$ be statements. Construct truth tables for the following statements. For each statement, state whether it is a tautology, a contradiction or neither.

(a) $(A \vee \sim B) \wedge (\sim A \wedge B)$.

(b) $(A \implies \sim B) \implies \sim B$.

(c) $(\sim A \wedge B) \implies (\sim B \wedge A)$.

(d) $(\sim A \wedge (B \vee \sim B)) \iff \sim A$.

**3.** (**23 marks**) For any integer $n$, let $A(n)$ be the statement:

"If $n = 3q - 1$ or $n = 3q - 2$ for some $q \in \mathbb{Z}$, then $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$."

(a) (**3 marks**) Write down the negation of $A(n)$.

(b) (**4 marks**) Write down the contrapositive of $A(n)$.

(c) (**3 marks**) Write down the converse of $A(n)$.

(d) (**5 marks**) Use a **direct proof** to show that $(\forall\, n \in \mathbb{Z})\ A(n)$.

(e) (**3 marks**) Is the contrapositive of $A(n)$ true for all $n \in \mathbb{N}$? Give brief reasons for your answer.

(f) (**5 marks**) Use **proof by contradiction** to show that the converse of $A(n)$ is true for all $n \in \mathbb{Z}$.

**4.** (**20 marks**) Let $A, B, C, D$ be sets. Define the **Cartesian product** (or simply the **product**) $A \times B$ of $A$ and $B$ by

$$A \times B := \{(a, b) : a \in A \quad \text{and} \quad b \in B\}.$$

(a) If $C$ is nonempty, then show that $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

(b) Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(c) Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(d) Show that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. Show also that in general,

$$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D).$$

**5.** (**6 marks**) Let $X = \{1, 9\}$ and let $Y = \{a, b, 9\}$. Find

(a) $\mathcal{P}(Y)$ and $\mathcal{P}(X \cap Y)$.

(b) $X \cup Y$ and $\mathcal{P}(X \cup Y)$.

**6.** (**9 marks**) Let $A = \{1, 2, \cdots, 10\}$.

(a) Given an example of a set $S$ such that $S \in \mathcal{P}(A)$ and the number of elements of $S$ is 4.

(b) Given an example of a set $S$ such that $S \subseteq \mathcal{P}(A)$ and the number of elements of $S$ is 4.

(c) Given an example of a two sets $S$ and $B$ such that $S \subseteq \mathcal{P}(A)$, the number of elements of $S$ is 4, $B \in S$ and the number of elements of $B$ is 3.