Monday: Functions as relations, one to one and onto functions

What is a function? [5.1]

Informally, a function from A to B is a rule which assigns to each element a of A a unique element $f(a)$ of B. Officially, we have

Definition. *A function f from A to B is a subset of $A \times B$ such that*

- *for each $a \in A$ there is a $b \in B$ with $(a, b) \in f$.*
- *if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.*

We write $f : A \to B$ to show that f is a function from A to B. If $a \in A$, we write $f(a)$ for the unique $b \in B$ such that $(a, b) \in f$.

Thus if f is a function, we have $f(a) = b \iff (a, b) \in f$. We will use this equivalence later.

Definition. *If $f : A \to B$, the A is the* domain *of f and B is the* codomain *of f. We write $\mathcal{D}om(f)$ for A and $\mathcal{C}odom(f)$ for B. We also define the* range *of f, $\mathcal{R}an(f)$, by*

$$\mathcal{R}an(f) = \{\, b \in B : (\exists a \in A)(f(a) = b)\,\}.$$

Note that $\mathcal{R}an(f) \subseteq \mathcal{C}odom(f)$, but there are examples where the two sets are not the same.

Definition (Equality of functions). *Two functions $f : A \to B$ and $g : A' \to B'$ are equal iff they are the same set of ordered pairs, in other words iff $A = A'$ and $f(a) = g(a)$ for all $a \in A$. [Notice that the textbook also requires that $B = B'$: most authors would consider that f and g are equal even if the codomains differ, as long as the domain and values are the same.]*

One-to-one and onto [5.1]

Definition. *A function $f : A \to B$ is* one-to-one *if for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$. It is* onto *if for each $b \in B$ there is at least one $a \in A$ with $f(a) = b$. It is a* one-to-one correspondence *or* bijection *if it is both one-to-one and onto.*

Notice that "f is one-to-one" is asserting uniqueness, while "f is onto" is asserting existence. This gives us the idea of how to prove that functions are one-to-one and how to prove they are onto.

Example 1. *Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 1$ is one-to-one and onto.*

Example 2. *Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = 2n + 1$ is one-to-one but not onto.*

For functions from $\mathbb{R}$ to $\mathbb{R}$, we can use the "horizontal line test" to see if a function is one-to-one and/or onto. The horizontal line $y = b$ crosses the graph of $y = f(x)$ at precisely the points where $f(x) = b$. So f is one-to-one if no horizontal line crosses the graph more than once, and onto if every horizontal line crosses the graph at least once.

Example 3. *Sketch graphs of the following functions $f : \mathbb{R} \to \mathbb{R}$ and determine whether each is one-to-one and/or onto.*

1. $f(x) = x^3 + x$.
2. $f(x) = x^3 - x$.
3. $f(x) = \mathrm{e}^x$.
4. $f(x) = x^2$.

Notice that in the definition of "onto", we need to know what the codomain is. So the function $f = \{ (x, \mathrm{e}^x) : x \in \mathbb{R} \}$ is not onto when thought of as a function from $\mathbb{R}$ to $\mathbb{R}$, but it is onto when thought of as a function from $\mathbb{R}$ to $(0, \infty)$.

Proposition 4. *Let $f : A \to B$ be a function. Then f is an onto function from A to $\mathcal{R}an(f)$. If f is one-to-one, then f is a bijection from A to $\mathcal{R}an(f)$.*

Tuesday: Composition of functions, Inverses

Composition of functions [5.2]

Definition. *Let $f : A \to B$ and $g : B \to C$. We define a new function $g \circ f : A \to C$ by declaring that $(g \circ f)(a) = g(f(a))$. We call $g \circ f$ "g* composed with *f".*

Example 5. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 2$ and $g(x) = 3x$. Find $g \circ f$ and $f \circ g$.*

The above example shows that $f \circ g$ and $g \circ f$ need not be equal. Of course, if A and C are not the same, they will not even be defined: if $g(b) \notin A$ then trying to figure out what $f(g(b))$ is gives a type error. [For example, "my mother's telephone number" makes sense but "my telephone number's mother" does not.

Composition of functions interacts with the notions of one-to-one and onto: it does preserve these properties, and in some cases if the composition has the property then so must the original functions.

Theorem 6. *Let $f : A \to B$ and $g : B \to C$ be functions.*

1. *If f and g are both one-to-one then $g \circ f$ is one-to-one.*
2. *If f and g are both onto then $g \circ f$ is onto.*
3. *If $g \circ f$ is one-to-one then f is one-to-one.*
4. *If $g \circ f$ is onto then g is onto.*

However there are examples of f and g with $g \circ f$ both one-to-one and onto but g not one-to-one and f not onto.

Although $\circ$ is not commutative, it is associative.

Theorem 7. *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ are functions then $(h \circ (g \circ f)) = ((h \circ g) \circ f)$.*

Proof. See pp 110 and 111 of the textbook, Problem 5.2.6.. □

Definition. *Let A be a set. The* identity function *on A, 1_A, is the function $1_A : A \to A$ given by $f(a) = a$ for all $a \in A$. In other words, $1_A = \{ (a, a) : a \in A \}$.*

Proposition 8. *Let $f : A \to B$ be a function. Then $1_B \circ f = f = f \circ 1_A$.*

Thursday: Inverses, images and preimages

Inverses [5.2]

Definition. *Let $f : A \to B$ be a function. An* inverse *of f is a function $g : B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.*

Example 9. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}x - \frac{1}{2}$. Then g is an inverse of f.*

Example 10. *Let $f : R \to [0, \infty)$ and $g : [0\infty) \to \mathbb{R}$ be given by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then g is* **not** *an inverse of f. Although we have $f(g(x)) = (\sqrt{x})^2 = x$, so $f \circ g = 1_{[0,\infty)}$, when we consider $g(f(x)) = \sqrt{x^2}$, we always get the positive square root, so for example $g(f(-2)) = \sqrt{(-2)^2} = \sqrt{4} = 2$.*

Lemma 11. *Let $f : A \to B$. Suppose g is an inverse of f. Then $g = \{ (f(a), a) : a \in A \}$.*

Proof. We must prove two inclusions: $g \subseteq \{ (f(a), a) : a \in A \}$ and $\{ (f(a), a) : a \in A \} \subseteq g$.

Let $p \in g$. Then $p = (b, g(b))$ for some $b \in B$. Now $f(g(b)) = b$, so we have $p = (f(g(b)), g(b))$, i.e. $p = (f(a), a)$ where $a = g(b)$. So $p \in \{ (f(a), a) : a \in A \}$ as required.

Conversely, let $a \in A$: we will show that $(f(a), a) \in g$. Now, we know that $g(f(a)) = a$. Since $g(y) = z \iff (y, z) \in g$, this means that $(f(a), a) \in g$, as required. □

Lemma 12. *Let $f : A \to B$ be a function. If f has an inverse, it is unique.*

Proof. Use Proposition 8 and Theorem 7. □

If f has an inverse, we write it as f^{-1}.

Theorem 13. *Let $f : A \to B$ be a function. Then f has an inverse if and only if f is a bijection.*

Proof. Suppose first that f has an inverse. We must show that f is one-to-one and onto.

One-to-one: Let $x, y \in A$ with $f(x) = f(y)$. Then $f^{-1}(f(x)) = f^{-1}(f(y))$, i.e. $x = y$, as required.

Onto: Let $b \in B$. Then $f(f^{-1}(b)) = b$, i.e. there is at least one $a \in A$ (namely $a = f^{-1}(b)$) such that $f(a) = b$.

Conversely, suppose that f is a bijection. We must find a candidate function g such that g is an inverse of f. From the earlier lemma, there is only one possible choice: we must have $g = \{ (f(a), a) : a \in A \}$. So we must check that this is indeed an inverse of f: we must show it is a function, that $f \circ g = 1_B$ and that $g \circ f = 1_A$.

Function: Let $b \in B$. On the one hand, since f is onto, there is at least one $a \in A$ with $f(a) = b$, so there is at least one $a \in A$ with $(f(a), a) = (b, a)$, so there is at least one $a \in A$ with $(b, a) \in g$. On the other hand, since f is one-to-one, there is at most one $a \in A$ with $f(a) = b$, so there is at most one $a \in A$ with $(b, a) \in g$. So g is indeed a function.

$f \circ g = 1_B$: Let $b \in B$. Put $a = g(b)$. So $(b, a) \in g$, so $(b, a) = (f(c), c)$ for some $c \in A$. We must have $c = a$, so $b = f(a)$, i.e. $b = f(g(b))$, as required.

$g \circ f = 1_A$: Let $a \in A$. Then $(f(a), a) \in g$, so $g(f(a)) = a$, as required.

□

Images and preimages [5.3]

Definition. *Let $f : A \to B$ be a function. For $S \subseteq B$ we define the* inverse image *or* preimage *of S under f to be*

$$f^{-1}(S) = \{\, a \in A : f(a) \in S \,\}.$$

Notice that $f^{-1}(S)$ is a **subset** of A, not an element of A. It is defined whether or not f^{-1} exists as a function. Note that there is a slight ambiguity here: if f happens to be a bijection then $f^{-1}(b)$ is an element of A for $b \in B$ and $f^{-1}(S)$ is a subset of A for $S \subseteq B$. Since the elements of B are not (usually) subsets of B, this ambiguity should never cause a problem.

Example 14. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find the following sets:*

1. $f^{-1}(\{4\})$.
2. $f^{-1}([-2, 9])$.
3. $f^{-1}((1, 4])$.
4. $f^{-1}(\{-9\})$.

To prove facts about preimages, we use the equivalence that

$$x \in f^{-1}(S) \iff f(x) \in S.$$

Example 15. *Let $f : A \to B$ be a function, $S, T \subseteq B$. Then*

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
2. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.
3. $f^{-1}(S^{\complement}_B) = f^{-1}(S)^{\complement}_A$.

In the same way as the preimage, we define the image of a subset of A:

Definition. *Let $f : A \to B$ be a function and $S \subseteq A$. We define the* image *of S under f to be*

$$f(S) = \{\, f(a) : a \in S \,\}.$$

Example 16. *Let $f : A \to B$ be a function, $S, T \subseteq B$. Then*

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
2. $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Notice that in the second of these, we only get $\subseteq$ rather than $=$: we can find examples with $f(S \cap T) \subset f(S) \cap f(T)$.

Friday: Orders and functions

Definition. *Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be sets. A function $f : A \to B$ is* order-preserving *if for all $x, y \in A$,*

$$x \preceq_A y \implies f(x) \preceq_B f(y).$$

It is strictly order preserving *if for all $x, y \in A$,*

$$x \preceq_A y \iff f(x) \preceq_B f(y).$$

For example, a constant function (in other words a function f such that there is some $b \in B$ with $f(x) = b$ for all $x \in A$) is order preserving, but is not strictly order preserving unless A is empty or has only one element.

Proposition 17. *If $(A, \preceq_A)$ and $(B, \preceq_B)$ are posets and $f : A \to B$ is strictly order preserving then f is one-to-one.*

Definition. *Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets. An* order-isomorphism *from A to B is a bijection $f : A \to B$ such that f and f^{-1} are both order-preserving. If there exists such an isomorphism, we say that A and B are* order-isomorphic.

Example 18. *Let $A = \{\, n \in \mathbb{N} : n \mid 30 \,\}$ and $B = \mathcal{P}(\{2, 3, 5\})$. Define $f : A \to B$ by $f(n) = \{\, m \in \{2, 3, 5\} \mid m \mid n \,\}$. Then f is an order-isomorphism from $(A, |)$ to $(B, \subseteq)$.*

Example 19. *Show that $(\mathbb{Z}, \leq)$ is order-isomorphic to $(E, \leq)$, where E is the set of even integers.*

Theorem 20. *Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets and $f : A \to B$ a bijection. Then f is an order-isomorphism iff f is strictly order preserving.*

Theorem 21. *Let A and B be posets and $f : A \to B$ an order-isomorphism. Then $x \in A$ is maximal in A iff $f(x)$ is maximal in B.*

Example 22. *Let $A = \{\, 1 - \frac{1}{n} : n \in \mathbb{N}\}$, and $B = A \cup \{1\}$, with the usual order they get as subsets of $\mathbb{R}$. Then $\mathbb{N}$ is order isomorphic to A but $\mathbb{N}$ is not order-isomorphic to B.*