Monday: Continuous functions

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let $a \in A$. Then f is* continuous at a *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. We say that f is* continuous *if it is continuous at a for all $a \in A$.*

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and let $a \in \mathbb{R}$. Then f is continuous at a.*

Proof. Let $\varepsilon > 0$. Put $\delta = \min\{1, \frac{\varepsilon}{2|a|+1}\}$. Let $x \in \mathbb{R}$ with $|x - a| < \delta$. Put $h = x - a$, so $x = a + x$. Then

$$\begin{aligned}
|f(x) - f(a)| &= |f(a + h) - f(a)| \\
&= |(a + h)^2 - a^2| \\
&= |a^2 + 2ah + h^2 - a^2| \\
&= |2ah + h^2| \\
&= |2a + h||h| \\
&\leq (|2a| + |h|)|h| \\
&\leq (2|a| + 1)|h| && \text{(since } |h| < 1) \\
&< (2|a| + 1)\delta \\
&= \varepsilon,
\end{aligned}$$

as required. □

Example 2. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$. Then f is not continuous at 0.*

Proof. Suppose for a contradiction that f is continuous at 0. Then, since $\frac{1}{2} > 0$, there is some $\delta > 0$ such that if $|x - 0| < \delta$ then $|f(x) - f(0)| < \frac{1}{2}$. Choose $n \in \mathbb{N}$ with $n > \frac{1}{2}\left(\frac{2}{\pi\delta} - 1\right)$. Then $2n+1 > \frac{2}{\pi\delta}$, so $\frac{(2n+1)\pi}{2} > \frac{1}{\delta}$, so $\frac{2}{(2n+1)\pi} < \delta$. Put $x = \frac{2}{(2n+1)\pi}$. Then $|x| < \delta$, so $|f(x)| < \frac{1}{2}$. However, $f(x) = \sin\left((2n+1)\frac{\pi}{2}\right)$, so $f(x) = \pm 1$, so $|f(x)| = 1 \not< \frac{1}{2}$. This contradiction shows that there is no such δ, and hence f is not continuous at 0. □

The intermediate value theorem

Theorem 3 (The intermediate value theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $k \in \mathbb{R}$ with $f(a) < k < f(b)$. Then there is some $c \in (a, b)$ with $f(c) = k$.*

Proof. Put $S = \{ x \in [a,b] : f(x) < k \}$. Then $a \in S$ so $S \neq \emptyset$, and S is bounded above by b, so S has a supremum. Put $c = \sup S$.

Claim: $f(c) \not< k$.

For: Suppose for a contradiction that $f(c) < k$. Put $\varepsilon = k - f(c)$, and choose $\delta > 0$ so that if $x \in [a,b]$ with $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Note that if $|f(x) - f(c)| < \varepsilon$ then $f(x) - f(c) < \varepsilon = k - f(c)$, so $f(x) < f(c)$. Thus $|b - c| \not< \delta$, so $c + \delta \leq b$. Put $x = c + \frac{\delta}{2}$. Then $x > c = \sup S$, so $x \notin S$. However, $f(x) < f(c) + \varepsilon = k$, and $x \in [a,b]$, so $x \in S$. This contradiction showws that we cannot have $f(c) < k$.

Claim: $f(c) \not> k$.

For: Suppose for a contradiction that $f(c) > k$. Put $\varepsilon = f(c) - k$. Choose $\delta > 0$ such that if $x \in [a,b]$ with $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Since $\delta > 0$ and $c = \sup S$, there is some $x \in S$ with $c - \delta < x \leq c$. But then $|x - c| < \delta$, so $|f(x) - f(c)| < \varepsilon$, so $f(x) - f(c) > -\varepsilon = -(f(c) - k) = k - f(c)$. Thus $f(x) > k$. But this contradicts the assumption that $x \in S$ so $f(x) < k$. Hence there is no such x and therefore we cannot have $f(c) > k$.

Thus we cannot have $f(c) < k$ or $f(c) > k$, so $f(c) = k$, as required. Finally, note that since $a \in S$ and b is an upper bound for S, $a \leq supS \leq b$, i.e. $a \leq c \leq b$. Since $f(a) \neq f(c) \neq f(b)$ we have $a \neq c \neq b$ so $a < c < b$, i.e. $c \in (a,b)$ as required. □

Tuesday: Continuity in terms of limits, open and closed sets and sequences

Limits of functions

Definition. *Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. We define the* ε-ball centred at a, *$B_\varepsilon(a)$, by*

$$B_\varepsilon(a) = \{ x \in \mathbb{R} : |x - a| < \varepsilon \},$$

and the deleted ε-ball centred at a, *$B'_\varepsilon(a)$, by $B'_\varepsilon(a) = B_\varepsilon(a) \setminus \{a\}$.*

Definition. *Let $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then a is a* limit point *of A if, for every $\varepsilon > 0$, $B_\varepsilon(a) \cap A \neq \emptyset$, and a is an* accumulation point *of A if for all $\varepsilon > 0$, $B'_\varepsilon(a) \cap A \neq \emptyset$.*

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let a be an accumulation point of A and let $L \in \mathbb{R}$. We say that $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.*

Notice the big difference between the definition of a limit and the definition of continuity: we insist that $0 < |x - a| < \delta$, in other words we do not test whether $|f(x) - L| < \varepsilon$ holds at $x = a$, only at values of x close to but not exactly equal to a. Thus, for example $\lim_{x \to 0} \frac{\sin x}{x}$ makes sense without having to explain that we never intend to evaluate $\frac{\sin 0}{0}$.

Example 4. *Define the function* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x$ *if* $x \notin \mathbb{Z}$, $f(x) = 0$ *if* $x \in \mathbb{Z}$. *Then* $\lim_{x\to 2} f(x) = 2$.

The two definitions, continuity and limits, fit together by the following result.

Theorem 5. *Let* $A \subseteq \mathbb{R}$ *and let* $f : A \to \mathbb{R}$ *be a function. Then* f *is continuous if and only if, for every* $a \in A$, *if* a *is an accumulation point of* A *then* $\lim_{x\to A} f(x) = f(a)$.

Proof. Exercise. □

Open and closed sets

Definition. *A subset* U *of* $\mathbb{R}$ *is* open *if for every* $x \in U$ *there is some* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq U$. *A subset* C *of* $\mathbb{R}$ *is* closed *if* $C^{\complement}_{\mathbb{R}}$ *is open.*

Proposition 6. *Let* $C \subseteq \mathbb{R}$. *Then* C *is closed if and only if, for every sequence* (s_n) *in* C, *if* $(s_n) \to a$ *as* $n \to \infty$ *then* $a \in C$.

Proof. Suppose first that C is closed. We must show that if (s_n) is a convergent sequence in C then the limit of the sequence is also in C. So suppose that $s_n \to a$ as $n \to \infty$. Suppose, for a contradiction that $a \notin C$. Then $a \in C^{\complement}$, and $C^{\complement}$ is open, so there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq C^{\complement}$. Since $s_n \to a$, there is an $N \in \mathbb{N}$ such that for $n > N$, $|s_n - a| < \varepsilon$. But then $s_{N+1} \in B_\varepsilon(a)$, so $s_{N+1} \in C^{\complement}$, contradicting the assumption that (s_n) is a sequence in C. So we cannot have $a \notin C$, so $a \in C$.

Conversely, suppose that for every sequence in C, if $s_n \to a$ then $a \in C$. Put $U = C^{\complement}$. We must show that U is open. So let $a \in U$. Suppose, for a contradiction, that there is no $\varepsilon > 0$ with $B_\varepsilon(a) \subseteq U$. In particular, for each $n \in \mathbb{N}$ we have $B_{\frac{1}{n}}(a) \nsubseteq U$, so there is some $s_n \in B_{\frac{1}{n}}(a) \setminus U$. But then $s_n \notin C^{\complement}$, so $s_n \in C$.

Claim: $s_n \to a$ as $n \to \infty$.

For: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then $\frac{1}{N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n > N$. Then $\frac{1}{n} < \frac{1}{N}$. Since $s_n \in B_{\frac{1}{n}}(a)$, $|s_n - a| < \frac{1}{n} < \frac{1}{N} < \varepsilon$, so $|s_n - a| < \varepsilon$ as required.

Thus (s_n) is a sequence in C which converges to a, but $a \notin C$, contradicting our assumption about C. □

Lemma 7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. For every open set* U, $f^{-1}(U)$ *is open.*

Proof. Let U be open, and let $a \in f^{-1}(U)$. Then $f(a) \in U$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq U$. By continuity, there is some $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Claim: $B_\delta(a) \subseteq f^{-1}(U)$.

For: Let $x \in B_\delta(a)$. Then $|x-a| < \delta$, so $|f(x) - f(a)| < \varepsilon$, so $f(x) \in B_\varepsilon(f(a)) \subseteq U$, so $f(x) \in U$, so $x \in f^{-1}(U)$, as required.

□

The converse is also true: to prove it, we first have to use the triangle inequality to prove that every ε-ball $B_\varepsilon(a)$ is open.

Lemma 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is continuous if and only if, for every sequence (s_n) in $\mathbb{R}$, if $s_n \to a$ as $n \to \infty$ then $f(s_n) \to f(a)$ as $n \to \infty$.*

Proof. Suppose first that f is continuous. Let (s_n) be a sequence in $\mathbb{R}$. Suppose $s_n \to a$ as $n \to \infty$. Let $\varepsilon > 0$. By continuity, there is some $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Since $s_n \to a$, there is some N such that if $n > N$ then $|s_n - a| < \delta$. Let $n > N$. Then $|s_n - a| < \delta$, so $|f(s_n) - f(a)| < \varepsilon$, as required.

We leave the converse as an exercise. □

Thursday: Differentiability

In today's lecture we will learn exactly what it means for a function to be differentiable. Before doing that, we will find a little more about limits.

Limits of products and quotients

Theorem 9. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ be functions, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist then $\lim_{x\to c} f(x)g(x)$ exists and is equal to $\lim_{x\to c} f(x) \lim_{x\to c} g(x)$.*

Proof. Let $F = \lim_{x\to c} f(x)$ and $G = \lim_{x\to c} g(x)$. Put $\eta = \frac{\varepsilon}{|G|+\frac{1}{2}+|F|}$. Choose $\delta_1\delta_2 > 0$ so that if $0 < |x-c| < \delta_1$ then $|f(x) - F| < \eta$ and if $x \in A$ with $0 < |x-c| < \delta_2$ then $|g(x) - G| < \min\{\eta, \frac{1}{2}\}$. Note that if $x \in A$ with $0 < |x-c| < \delta_2$ then $|g(x) - G| < \frac{1}{2}$ so $|g(x) < |G| + \frac{1}{2}$. Put $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in A$ with $1 < |x - c| < \delta$. Then

$$\begin{aligned}
|f(x)g(x) - FG| &= |f(x)g(x) - Fg(x) + Fg(x) - FG| \\
&\leq |f(x)g(x) - Fg(x)| + |Fg(x) - FG| && \text{(triangle inequality)} \\
&= |f(x) - F||g(x)| + |F||g(x) - G| \\
&\leq \eta(|G| + \frac{1}{2}) + |F|\eta \\
&= \varepsilon
\end{aligned}$$

Thus $\lim_{x\to c} f(x)g(x)$ exists and equals FG. □

Note that we might be a little lazy and write this as "$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$". However, we must remember that limits need not exist, and the existence of the limit is part of the assertion. Also the converse does not hold: it is quite possible for $\lim_{x\to c} f(x)g(x)$ but neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ to exist.

Theorem 10. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ exists and is non-zero then $\lim_{x\to c} \frac{1}{f(x)}$ exists and is equal to $\frac{1}{\lim_{x\to c} f(x)}$.*

Proof. Exercise. Note that we need to choose δ small enough to ensure that $f(x)$ is non-zero within a distance of δ from c: in fact we will want to ensure that $\frac{1}{f(x)}$ does not get too large—say, does not get larger than $2F$ where $F = \lim_{x\to c} f(x)$—so we will choose δ small enough to ensure that $|f(x) - F| < \frac{F}{2}$, which ensures that $|f(x)| > |F - \frac{F}{2}|$. See the proof that if $b_n \to B \neq 0$ then $\frac{1}{b_n} \to \frac{1}{B}$ in the notes for week 11 for more ideas. □

Exercise 11. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ be functions, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist then $\lim_{x\to c} f(x) + g(x)$ exists and is equal to $\lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.*

Differentiability

Definition. *Let $A \subseteq \mathbb{R}$. If $c \in A$, we say that c is an* interior point *of A if there is some $\varepsilon > 0$ such that $B_\varepsilon(c) \subseteq A$. We denote the set of interior points of A by* $\operatorname{int}(A)$.

Thus A is open if and only if every point of A is an interior point of A, i.e. if $A = \operatorname{int}(A)$.

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $c \in \operatorname{int}(A)$. We say that f is* differentiable at c *if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, or equivalently if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. If the limit exists, we denote it by $f'(c)$, and call this number the* derivative of f at c. *For $S \subseteq \operatorname{int}(A)$ we say that f is* differentiable on S *if f is differentiable at all $c \in S$. When A is open we say that f is* differentiable *if it is differentiable on A.*

Example 12. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is differentiable and, for all $c \in \mathbb{R}$, $f'(c) = 2c$.*

Proof. For all $h \neq 0$ we have

$$\begin{aligned} \frac{f(c+h) - f(c)}{h} &= \frac{(c+h)^2 - c^2}{h} \\ &= \frac{c^2 + 2ch + h^2 - c^2}{h} \\ &= \frac{2ch + h}{h} \\ &= 2c + h \end{aligned}$$

Now $\lim_{h\to 0} 2c + h = 2c$, so $f'(c)$ exists and equals $2c$, as required. □

Theorem 13. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $c \in \operatorname{int}(A)$. If f is differentiable at c then f is continuous at c.*

Proof. Suppose f is differentiable at c. Then $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists and equals $f'(c)$. We also have $\lim_{x\to c}(x-c)$ exists and equals 0. So by Theorem 9, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c)$ exists and equals $f'(c)\cdot 0 = 0$. So $\lim_{x\to c}(f(x)-f(c) = 0$, so $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c. □

Friday: Rolle's Theorem and the Mean Value Theorem

Lemma 14. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains its bounds. In other words, there exist $c, d \in [a, b]$ with $f(c) = \max\{\, f(x) : x \in [a, b] \,\}$ and $f(d) = \min\{\, f(x) : x \in [a, b] \,\}$.*

Proof. First we will show that f is bounded. Suppose, for a contradiction, that f is unbounded above. For each $n \in \mathbb{N}$ we can find some $x_n \in [a, b]$ with $f(x_n) > n$. Now the sequence (x_n) is bounded, so it has a convergent subsequence, (x_{i_n}) say. Let x be the limit of this subsequence. Since (x_n) is a sequence in $[a, b]$, which is closed, we have $x \in [a, b]$. But then $f(x_{i_n})$ converges to $f(x)$, which is impossible because $(f(x_{i_n}))$ is an unbounded sequence.

Similarly, f is bounded below.

Put $s = \sup\{\, f(x) : x \in [a, b] \,\}$. For every n, there is some $y_n \in [a, b]$ with $s - \frac{1}{n} < f(y_n) \le s$. Then (y_n) is a bounded sequence in $[a, b]$, so it has a subsequence (y_{j_n}) which converges to some $c \in [a, b]$. By continuity, $(f(y_{j,n}))$ converges to $f(c)$. But, by construction, $(f(y_{j_n})$ converges to s. So $f(c) = s = \sup\{\, f(x) : x \in [a, b] \,\}$. So $f(c) = \max\{\, f(x) : x \in [a, b] \,\}$.

Similarly, f attains its infimum. □

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Then a is a* local maximum *of f if there is some $\varepsilon > 0$ such that for all $x \in A$ with $|x - a| < \varepsilon$ we have $f(x) \le f(a)$. Similarly, a is a* local minimum *of f if there is some $\varepsilon > 0$ such that for all $x \in A$ with $|x - a| < \varepsilon$ we have $f(x) \ge f(a)$.*

Theorem 15. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $a \in \operatorname{int} A$. If $f'(a)$ exists and a is a local maximum or local minimum of f then $f'(a) = 0$.*

Proof. Exercise. □

Note that we need both $f'(a)$ exists and $a \in \operatorname{int}(A)$ as hypotheses here: consider the examples $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x$, which has 1 as a local maximum, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |x|$ which has 0 as a local minimum.

Theorem 16 (Rolle's Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there is some $c \in (a, b)$ with $f'(c) = 0$.*

Proof. Put $k = f(a) = f(b)$. We know that f is continuous on $[a, b]$ so it attains its maximum at some $c \in [a, b]$. Suppose first that $c \neq a$ and $c \neq b$. Then $c \in (a, b)$, so c is a local maximum of f, and $f'(c)$ exists, so by the previous result we must have $f'(c) = 0$.

Similarly, we know that f attains its minimum at some $d \in [a, b]$, and if $d \neq a, b$ then $d \in (a, b)$ and $f'(d) = 0$.

The only remaining possibility is that $c, d \in \{a, b\}$. But then we must have $f(x) = k$ for all $x \in [a, b]$, so $f'(x) = 0$ for all $x \in (a, b)$. □

Theorem 17 (Mean Value Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Then there is some $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Put $k = \frac{f(b)-f(a)}{b-a}$ and define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - kx$. Then g is continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) = f'(x) - k$. Also

$$\begin{aligned} g(b) - g(a) &= f(b) - kb - f(a) - ka \\ &= (f(b) - f(a)) - k(b - a) \\ &= (f(b) - f(a)) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= 0, \end{aligned}$$

so $g(a) = g(b)$. Hence by Rolle's Theorem there is some $c \in (a, b)$ with $g'(c) = 0$. But then $f'(c) - k = 0$, so $f'(c) = k$, as required. □