Monday: Completeness

The ordered field axioms are not yet enough to characterise the real numbers, as there are other examples of ordered fields besides the real numbers. The most familiar of these is the set of rational numbers. The final axiom we give is the completeness axiom, which is satisfied by $\mathbb{R}$ but not by $\mathbb{Q}$.

Definition. *A* complete *ordered field is an ordered field F with the least upper bound property (in other words, with the property that if $S \subseteq F$, $S \neq \emptyset$ and S is bounded above then S has a least upper bound* $\sup S$*).*

Example 1. *The real numbers are a complete ordered field.*

We will see in a moment that the rational numbers are not complete.

Lemma 2. *Let F be a complete ordered field, and let $S \subseteq F$, $x \in F$. Then the following are equivalent:*

- $x = \sup S$
- *x is an upper bound for S and, for each $\varepsilon \in F$ with $\varepsilon > 0_F$ there is some $s \in S$ with $x - \varepsilon < s \leq x$.*

Proposition 3 (The Archimedean property of $\mathbb{R}$). *For every $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $n > x$.*

Proof. Let $x \in \mathbb{R}$. Suppose, for a contradiction, that there is no $n \in \mathbb{N}$ with $n > x$. Then, since $\leq$ is a total order, we have $n \leq x$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}$ is bounded above. We also have $\mathbb{N} \neq \emptyset$, so $\mathbb{N}$ must have a least upper bound, s. Since $s = \sup \mathbb{N}$ and $1 > 0$, there is some $n \in \mathbb{N}$ with $s - 1 < n \leq s$. But then $s < n + 1$, so $n + 1 \not\leq s$. However, $n + 1 \in \mathbb{N}$ and s is an upper bound for $\mathbb{N}$ so $n + 1 \leq s$. This contradiction shows that there must be some $n \in \mathbb{N}$ with $n > x$, as required. □

Proposition 4. *There is some real number a with $a^2 = 2$.*

Proof. Let $S = \{ x \in \mathbb{R} : x^2 < 2 \}$.

Claim: $S \neq \emptyset$.

For: $0^2 = 0 < 2$, so $0 \in S$.

Claim: S is bounded above.

For: We will show that 2 is an upper bound for S. So, let $x \in S$. Suppose, for a contradiction, that $x \not\leq 2$. Then $x > 2$, so $x^2 > 2^2 = 4 > 2$, so $x^2 \not< 2$, contradicting the assumption that $x \in S$.

From this we know that S has a least upper bound. Put $a = \sup S$: we will show that $a^2 = 2$ as required.

Claim: $a^2 \not< 2$.

For: Suppose, for a contradiction, that $a^2 < 2$. Put $p = 2 - a^2$ and $\varepsilon = \frac{p}{5}$. Notice that $a \leq 2$, because 2 is an upper bound for S, and $0 < p \leq 2$ so $\varepsilon < 1$

$$\begin{aligned}(a+\varepsilon)^2 &= a^2 + 2a\varepsilon + \varepsilon^2 \\ &< a^2 + 2\cdot 2\cdot \varepsilon + 1\cdot\varepsilon && \text{(since } a \leq 2 \text{ and } \varepsilon < 1) \\ &= a^2 + 5\varepsilon \\ &= 2.\end{aligned}$$

So we have $(a+\varepsilon)^2 < 2$, so $a + \varepsilon \in S$. But $a < a + \varepsilon$, contradicting the fact that a is an upper bound for S. Thus we cannot have $a^2 < 2$.

Claim: $a^2 \not> 2$.

For: Suppose, for a contradiction, that $a^2 > 2$. Put $r = a^2 - 2$, and $\varepsilon = \frac{r}{4}$. Then $\varepsilon > 0$, so since $a = \sup S$ there is some $s \in S$ with $a - \varepsilon < s \leq a$. Since $s > a - \varepsilon$ we have

$$\begin{aligned}s^2 &> (a-\varepsilon)^2 \\ &= a^2 - 2a\varepsilon + \varepsilon^2 \\ &\geq a^2 - 4\varepsilon && \text{(since } \varepsilon^2 \geq 0) \\ &= 2,\end{aligned}$$

so $s^2 > 2$, contradicting the assumption that $s \in S$. This shows that we cannot have $a^2 > 2$.

Hence we must have $a^2 = 2$, as required. □

Tuesday: Sequences

Sequences [5.5, 8.5]

Definition. *Let A be a set. A* sequence in A *is a function $s : \mathbb{N} \to A$. We usually write $s(n)$ as s_n, and we write (s_n) or $s_1, s_2, s_3, \ldots$ for the whole sequence.*

Example 5. *The sequence $(\frac{n-1}{n})$ has $s_n = \frac{n-1}{n}$, so it is the sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots$.*

Definition. *Let (s_n) be a sequence in $\mathbb{R}$. We say that (s_n) is*

- increasing *if for all $n \in \mathbb{N}$, $s_n \leq s_{n+1}$;*
- strictly increasing *if for all $n \in \mathbb{N}$, $s_n < s_{n+1}$;*

- decreasing *if for all* $n \in \mathbb{N}$, $s_n \geq s_{n+1}$;
- strictly decreasing *if for all* $n \in \mathbb{N}$, $s_n > s_{n+1}$;
- monotonic *if it is either increasing or decreasing;*
- bounded above *if* $\{ s_n : n \in \mathbb{N} \}$ *is bounded above;*
- bounded below *if* $\{ s_n : n \in \mathbb{N} \}$ *is bounded below; and*
- bounded *if it is both bounded above and below.*

Example 6. *The sequence* $(\frac{n-1}{n})$ *is strictly increasing (si it is increasing, so it is monotone), and is bounded above by* 1 *and below by* 0*, so it is bounded.*

Definition. *For* $a \in \mathbb{R}$ *we define* $|a|$ *by*

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{otherwise.} \end{cases}$$

Thus we have $|a| \geq 0$ *for all* $a \in \mathbb{R}$*, with* $|a| > 0$ *unless* $a = 0$*.*

Proposition 7. *For any* $a, x \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$ *with* $\varepsilon > 0$*, we have* $|a - x| < \varepsilon$ *iff* $a - \varepsilon < x < a + \varepsilon$*.*

Proof. Exercise. □

Definition. *Let* (s_n) *be a sequence in* $\mathbb{R}$*, and let* $L \in \mathbb{R}$*. We say that* (s_n) converges to L *if for every* $\varepsilon \in \mathbb{R}$ *with* $\varepsilon > 0$ *we can find an* $N \in \mathbb{N}$ *such that for all* $n > N$, $|s_n - L| < \varepsilon$*. If* (s_n) *converges to* L*, we write* $s_n \to L$ *as* $n \to \infty$*, and call* L *a* limit *of the sequence* (s_n)*.*

Example 8. *The sequence* $(\frac{n-1}{n})$ *converges to* 1*.*

Example 9. *The sequence* $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ *converges to* 0*.*

Theorem 10. *If the sequence* (s_n) *in* $\mathbb{R}$ *has a limit, then the limit is unique.*

Thursday: Subsequences and monotonic sequences

Subsequences [5.5]

A *subsequence* of a sequence (s_n) is a sequence formed by taking certain terms from the original sequence, in the same order as they appeared in the original sequence. For example, if we have the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ then we may form the subsequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$. More precisely, we have the following definition.

Definition. *A* subsequence *of a sequence* (s_n) *is a sequence* (s_{i_n})*, where* (i_n) *is a strictly increasing sequence in* $\mathbb{N}$*.*

Lemma 11. *If* (i_n) *is a strictly increasing sequence in* $\mathbb{N}$ *then for all* $n \leq i_n$, $n \leq i_n$*.*

Proof. Exercise. □

Proposition 12. *Let (s_n) be a sequence in $\mathbb{R}$, and (s_{i_n}) a subsequence of (s_n). If $s_n \to L$ as $n \to \infty$ then $s_{i_n} \to L$ as $n \to \infty$.*

Proof. Suppose $s_n \to L$ as $n \to \infty$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - L| < \varepsilon$. Now let $n > N$. Then $i_n \geq n > N$, so $i_n > N$, so $|s_{i_n} - L| < \varepsilon$. □

Theorem 13. *Let (s_n) be a monotonic bounded sequence in $\mathbb{R}$. Then (s_n) converges to some $L \in \mathbb{R}$*

Proof. Suppose first that (s_n) is increasing. The set $S = \{ s_n : n \in \mathbb{N} \}$ is non-empty (since $s_1 \in S$) and bounded above, so it has a least upper bound, L say. We claim that $s_n \to L$ as $n \to \infty$. So let $\varepsilon > 0$. Then there is some $s \in S$ with $L - \varepsilon < s \leq L$. Now, $s \in S$ so $s = s_N$ for some $N \in \mathbb{N}$. Let $n > N$. Then $s_N \leq s_n$, since (s_n) is increasing, so we have $L - \varepsilon < s_N \leq s_n \leq L < L + \varepsilon$, so $L - \varepsilon < s_n < L + \varepsilon$, so $|s_n - L| < \varepsilon$. Thus $s_n \to L$, as claimed.

We leave the case when (s_n) is a decreasing sequence as an exercise. □

Theorem 14. *Let (s_n) be a sequence in $\mathbb{R}$. Then (s_n) has a subsequence which is monotonic.*

The idea is as follows: we give a method for constructing an increasing subsequence in (s_n), which will work unless some particular thing goes wrong. We will then give an alternative method which gives a decreasing subsequence, and which will work if that particular thing went wrong with the first method.

Lemma 15. *Let (s_n) be a sequence in $\mathbb{R}$ with no greatest term. Then (s_n) has an increasing subsequence.*

Proof. We construct the subsequence (s_{i_n}) recursively. The sequence has the property that

$$\text{for all } j, k \in \mathbb{N}, \text{ if } j \leq i_k \text{ then } s_j \leq s_{i_k}. \qquad (*)$$

First we let $i_1 = 1$. This certainly satisfies $(*)$ since there is no j with $j < 1$. Now suppose we have chosen $i_1 < i_2 < \cdots < i_n$ satisfying $(*)$. We know that s_{i_n} is not the greatest term in the sequence, since there is no greatest term, so there is some m with $s_{i_n} < s_m$. However, $s_j \leq s_{i_n}$ for all $j \leq i_n$, so if $s_{i_n} < s_m$ then $m > i_n$. We let i_{n+1} be the least $m > i_n$ with $s_{i_n} \leq s_m$. We must check that this choice also satisfies $(*)$. We have assumed that it is satisfied for all i_ks for $k \leq n$, so we only need to check it for i_{n+1}. So suppose $j < i_{n+1}$. If $j \leq i_n$ then $s_j \leq s_{i_n} \leq s_{i_{n+1}}$. If $i_n < j < i_{n+1}$ then, since i_{n+1} was the least m with $s_{i_n} \leq s_m$, we must have $s_j < s_{i_n} \leq s_{i_{n+1}}$.

Clearly, the subsequence (s_{i_n}) we have constructed is an increasing sequence, as required. □

Proof of Theorem 14. Let (s_n) be a sequence in $\mathbb{R}$. There are two possibilities: either there is an $n \in \mathbb{N}$ such that $\{ s_m : m > n \}$ has no greatest element, or there is no such n. In the latter case, for every $n \in \mathbb{N}$, $\{ s_m : m > n \}$ has a greatest element.

Case 1: Suppose there is some n_0 such that $\{ s_m : m > n_0 \}$ has no greatest element. For each k, put $t_k = s_{n_0+k}$. Then (t_k) has no greatest element, so by the previous lemma it has an increasing subsequence (t_{i_k}). But then $(s_{n_0+i_k})$ is an increasing subsequence of (s_n).

Case 2: Suppose that for every $n \in \mathbb{N}$, $\{ s_m : m > n \}$ has a greatest element. Recursively choose a subsequence of (s_n) as follows: i_1 is chosen so that $s_{i_1} \geq s_m$ for all $m > 1$, and once $i_1 < i_2 < \cdots < i_n$ have been chosen, i_{n+1} is chosen so that $i_n < i_{n+1}$ and $s_{i_{n+1}} \geq s_m$ for all $m > n$. Since $\{ s_m : m > n \}$ always has a greatest element, we can always find such i_1 and i_{n+1}. It remains only to show that this gives a decreasing subsequence. Note that for each n we have that s_{i_n} is the greatest element of $\{ s_m : m > k \}$ for some $k < i_n$, so $s_{i_n} \geq s_m$ for all $m > k$. In particular, since $k < i_n < i_{n+1}$, $s_{i_n} \geq s_{i_{n+1}}$ as required.

□

Friday: Cauchy sequences

We know what it means to say that (s_n) converges to L. To say that (s_n) converges means that (s_n) converges to some L, i.e.

$$(\exists L)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|s_n - L| < \varepsilon).$$

This is rather complicated: it has an extra layer of complexity with the extra change between $\exists$ and $\forall$ quantifiers. It is also awkward to check, since we have to find the limit L before we can check that the condition holds. An alternative property, which only mentions the sequence itself and not its possible limit, is the "Cauchy convergence criterion":

Definition. *A sequence (s_n) in $\mathbb{R}$ is a* Cauchy *sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|s_m - s_n| < \varepsilon$.*

We will prove that a sequence (s_n) in $\mathbb{R}$ converges if and only if it is a Cauchy sequence.

Lemma 16 (The Triangle Inequality). *Let $a, b \in \mathbb{R}$. Then $|a + b| \leq |a| + |b|$, and hence, if $x, y, z \in \mathbb{R}$ then $|x - z| \leq |x - y| + |y - z|$.*

Proof. Exercise. □

Proposition 17. *Let (s_n) be a sequence in $\mathbb{R}$. If (s_n) converges then (s_n) is bounded.*

Proof. Suppose $s_n \to L$ as $n \to \infty$. Putting $\varepsilon = \frac{1}{2}$, we know that there is some $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - L| < \frac{1}{2}$. So, for $n > N$ we have

$$|s_n| = |(s_n - L) + L| \leq |s_n - L| + |L| < |L| + \tfrac{1}{2}.$$

Thus for every n we have $|s_n| \leq \max\{|s_1|, |s_2|, \ldots, |s_N|, |L| + \frac{1}{2}\}$. So (s_n) is bounded. □

Lemma 18. *Let (s_n) be a bounded sequence. Then (s_n) has a convergent subsequence.*

Proof. We know that any sequence in $\mathbb{R}$ has a monotonic subsequence, and any subsequence of a bounded sequence is clearly bounded, so (s_n) has a bounded monotonic subsequence. But every bounded monotonic sequence converges. So (s_n) has a convergent subsequence, as required. □

Lemma 19. *Let (s_n) be a Cauchy sequence in $\mathbb{R}$. If (s_n) has a convergent subsequence then (s_n) converges.*

Proof. Let (s_{i_n}) be a subsequence which converges to L. Let $\varepsilon > 0$. Put $\eta = \varepsilon/2$. Choose N_1 so that if $m, n > N_1$ then $|s_m - s_n| < \eta$, choose N_2 so that if $n > N_2$ then $|s_{i_n} - L| < \eta$, and choose k so that $k > N_2$ and $i_k > N_1$ (for example, we may take $k = \max\{N_1 + 1, N_2 + 1\}$: certainly $k > N_2$ and $i_k \geq k > N_1$). Put $N = N_1$. Then

$$
\begin{aligned}
|s_n - L| &= |s_n - s_{i_k} + s_{i_k} - L| \\
&\leq |s_n - s_{i_k}| + |s_{i_k} - L| && \text{(triangle inequality)} \\
&< \eta + |s_{i_k} - L| && \text{(since } n, i_k > N_1) \\
&< \eta + \eta && \text{(since } k > N_2) \\
&= \varepsilon.
\end{aligned}
$$

Thus $|s_n - L| < \varepsilon$ as required. So (s_n) converges to L. □

Lemma 20. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Take $\epsilon = 1$ in the definition. Then there exists N such that $|a_m - a_n| < 1$ for all $m, n > N$. In particular $|a_m| = |a_m - a_{N+1} + a_{N+1}| < |a_m - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}|$ (since $N + 1 > N$). So if $K = \max\{|a_1|, |a_2|, \ldots, |a_N|, |a_{N+1} + 1|\}$ then $|a_m| \leq K$ for all m. □

Lemma 21. *Every convergent sequence in $\mathbb{R}$ is Cauchy.*

Proof. Exercise. □

Putting these results together gives our main result:

Theorem 22. *A sequence in $\mathbb{R}$ is a Cauchy sequence if and only if it converges.*

Limits of sums and products

Theorem 23. *Let (a_n), (b_n) be sequences in $\mathbb{R}$. Suppose that $a_n \to A$ and $b_n \to B$ as $n \to \infty$. Then*

1. *$a_n + b_n \to A + B$ as $n \to \infty$;*
2. *$a_n b_n \to AB$ as $n \to \infty$; and*
3. *if $b_n \neq 0$ for all n and $B \neq 0$ then $\frac{a_n}{b_n} \to \frac{A}{B}$ as $n \to \infty$.*

Proof. For (1), let $\varepsilon > 0$. Put $\eta = \varepsilon/2$. Choose $N_1, N_2 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - A| < \eta$ and if $n > N_2$ then $|b_n - B| < \eta$. Put $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| && \text{(triangle inequality)} \\ &< \eta + \eta && \text{(since } n > N_1 \text{ and } n > N_2\text{)} \\ &= \varepsilon. \end{aligned}$$

so $a_n + b_n \to A + B$ as $n \to \infty$.

For (2), let $\varepsilon > 0$. Since (b_n) converges, it is bounded, so there is some $P > 0$ with $|b_n| < P$ for all n. Put $\eta = \frac{\varepsilon}{|A|+P}$. Choose $N_1, N_2 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - A| < \eta$ and if $n > N_2$ then $|b_n - B| < \eta$. Put $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - Ab_n + Ab_n - AB| \\ &\leq |a_n b_n - Ab_n| + |Ab_n - AB| && \text{(triangle inequality)} \\ &= |a_n - A||b_n| + |A||b_n - B| \\ &= |a_n - A|P + |A||b_n - B| \\ &< \eta P + |A|\eta \\ &= \varepsilon \end{aligned}$$

Thus $a_n b_n \to AB$ as $n \to \infty$.

For (3), we will first prove that $\frac{1}{b_n} \to \frac{1}{B}$ and then apply 2. So let $\varepsilon > 0$. Put $\eta = \frac{|B|^2\varepsilon}{2}$. Since $B \neq 0$, $\frac{|B|}{2} > 0$, so there is some N_1 such that if $n > N_1$ then $|b_n - B| < \frac{|B|}{2}$. Note that if $n > N_1$ then $|b_n| > |B| - \frac{|B|}{2} = \frac{|B|}{2}$, so $\left|\frac{1}{b_n}\right| < \frac{2}{|B|}$. Choose $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n - B| < \eta$. Put $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned} \left|\frac{1}{b_n} - \frac{1}{B}\right| &= \left|\frac{B - b_n}{b_n B}\right| \\ &= \left|\frac{1}{b_n}\right|\left|\frac{1}{B}\right||B - b_n| \\ &< \frac{2}{|B|}\frac{1}{|B|}|b_n - B| \\ &< \frac{2}{|B|^2}\eta \\ &= \varepsilon, \end{aligned}$$

so $\frac{1}{b_n} \to \frac{1}{B}$ as $n \to \infty$. The result then follows by (2). $\square$