Monday: Introduction

The motivation for this course is to learn the skill of *justifying* answers that we find. We will learn to give proofs, in other words to convince someone else that what we believe really is true.

Example: our method for finding $\int_1^2 \frac{1}{x^2}\,dx$ gives us a good answer, but the method fails when we use it to try to find $\int_{-1}^1 \frac{1}{x^2}\,dx$ (the answer should not be negative). So, when we use the method to answer problems such as the first, we should not simply go through the calculation: we should also justify our use of the method.

Read Chapter 0 of *Chapter Zero* for more background and motivation.

Statements and Predicates [1.2]

A *statement* is a sentence which is either true or false, e.g.

- $2 + 2 = 4$.
- George Bush is the President of the USA.
- $2 + 3 = 7$.

Examples of things which are not statements:

- What time is it?
- He is 1.9 metres tall.
- $n + 3 = 2$.

The last two fail because we need to specify who "He" is and what value n is. "He" and n are *free variables* and a sentence containing fre variables is a *predicate.*

- Picasso's *Les Demoiselles d'Avignon* is a beautiful painting.

This one fails because there is not a clear unambiguous definition of what makes a painting beautiful.

Another point on definitions: we must be clear and precise when we make definitions. For example, odd and even numbers. An integer n is *even* if $n = 2k$ for some integer k, but there are two possible definitions for *odd*: n is odd if n is not even, or n is odd if $n = 2k + 1$ for some integer k. We should be careful when we are giving a proof that the definition we are using for "odd" is the same as the definition being used by the person we are trying to convince. NB for now we wil just assume these definitions are equivalent: we will give a proof of this much later in the course.

Tuesday: Quantifiers and implications

Quantification [1.3, 1.4]

We can turn a predicate such as "$n + 3 = 5$" into a statement by *quantifying*, i.e. by saying for how many values of n this is true:

- For all natural numbers n, $n + 3 = 5$.
- There exists a natural number n such that $n + 3 = 5$.

Of course the first is false and the second is true. We sometimes express the second type of quatification as "For some natural number n, $n + 3 = 5$." When we say "For some" or "There exists" we don't suggest there is only one. If we want to say that, we would say either "There exists a unique natural number n such that $n + 3 = 5$." Symobls: we use $\forall$ as a symbol for "for all" and $\exists$ for "there exists ... such that".

Order is important: compare the two statements

- $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y = x^2)$.
- $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(y = x^2)$.

Often we are supposed to prove a statement such as "If n is even then n^2 is even. This is a predicate, not a statement. When this happens there is a hidden universal quatification assumed. In other words, we are supposed to prove that the statement "For all n, if n is even then n^2 is even" is true.

Implication [1.4, 1.5]

An implication is a statement in the form "If A then B", or "A implies B". A is called the *hypothesis* and B is called the *conclusion*. We sumbolise it as $A \implies B$.

The implication is true if A and B are both true, or if A is false (regardless of whether B is true or false). It is false when A is true and B is false. To see why we make this convention, consider the statement "For all integers n, if $n > 4$ then $n^2 > 8$." Try substituting different values of n in: $n = 1$, $n = 3$, $n = 5$.

When an implication $A \implies B$ is true because A is false, we sometimes say it is *vacuously true*.

The *converse* of an implication $A \implies B$ is the implication $B \implies A$. This might or might not be true, regardless of whether the original implication was true.

Wednesday: Compound statements and truth tables

Compound statements and truth tables [1.6]

We have other connectives besides "implies". We symbolise "A and B" by $A \wedge B$, "A or B" by $A \vee B$, "Not A" by $\neg A$ and "A if and only if B" by $A \iff B$. Note that $\vee$ means *inclusive or*, in other words "A or B, or both".

Statements built up using these connectives are *compound statements*. We can find their *truth values* (i.e. true or false) using *truth tables*

A	B	$A \implies B$	$A \wedge B$	$A \vee B$	$\neg A$	$A \iff B$
T	T	T	T	T	F	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example: construct the truth table for the statement $(A \wedge \neg B) \iff \neg(A \implies B)$.

A compound statement which is true, no matter what the truth values of the statements it is built from, is called a *tautology*. One which is always false is called a *contradiction*.

Thursday: Negation of statments and quantifiers

Negating statements [1.8]

When we negate a compound statement or a quantified statement, we change the connectives and the quantifiers:

- $\neg(A \wedge B)$ is the same as $\neg A \vee \neg B$.
- $\neg(A \vee B)$ is the same as $\neg A \wedge \neg B$.
- $\neg(A \implies B)$ is the same as $A \wedge \neg B$.
- $\neg(\forall x)A(x)$ is the same as $(\exists x)\neg A(x)$.
- $\neg(\exists x)A(x)$ is the same as $(\forall x)\neg A(x)$.

Exercise: justify all these with truth tables.

Examples: negate the following statements and predicates:

- Every natural number is either even or odd.
- There is an even number n such that n^2 is odd.
- If x is an odd integer then x^2 is an even integer.
- Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

Example: a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at x if for every $\epsilon > 0$ there is a $\delta > 0$ such that, for every y with $|y - x| < \delta$ we have $|f(x) - f(x)| < \epsilon$. What does it mean to say that f is not continuous at x?

Friday: Proving implications

Direct proof [1.12]

To give a direct proof of an implication $A \implies B$, we suppose that A is true, and use a series of steps to deduce that B must also be true.

Example: give a direct proof that, for any integer n, if n is even then n^2 is even.

Example: what goes wrong when we try to use a direct proof to show that for any integer n, if n^2 is even then n is even?

Proof by contraposition [1.13]

Exercise: show that $(A \implies B) \iff (\neg B \implies \neg A)$ is a tautology.

So we know that $A \implies B$ is true exactly whenever $\neg B \implies \neg A$ is true. The implication $\neg B \implies \neg A$ is called the *contrapositive* of the implication $A \to B$. To give a *proof by contraposition* of $A \implies B$, we give a direct proof of the contrapositive, i.e. we assume that B is false and we deduce that A must also be false.

Example: use proof by contraposition to show that for any integer n, if n^2 is even then n is even.

Proof by contradiction

On the assignment you are supposed to prove that $(A \implies B) \iff \neg(A \wedge \neg B)$ is a tautology. So, to show that $A \implies B$ is true, we can show taht $A \wedge \neg B$ must be false, which we do by assuming both A and $\neg B$, and arriving at some contradictory statement.

Example: use proof by contradiction to show that if a is a real number with $a > 0$ then $1/a > 0$.

Example: use proof by contradiction to show that $\sqrt{2}$ is irrational.