THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2003
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: **THREE** hours)

NOTE: This is an OPEN BOOK examination.
Answer ALL **EIGHT** questions. All questions carry equal marks.

1. For each $n \in \mathbb{N}$ let $A(n)$ be the implication "if n is even then $3n$ is even".
   (a) Write down the converse of $A(n)$. (2 marks)
   (b) Write down the contrapositive of $A(n)$. (2 marks)
   (c) Write down the negation of $A(n)$. (2 marks)
   (d) Use a **direct proof** to prove that $A(n)$ is true for all n. (4 marks)
   (e) Use **proof by contraposition** to prove that the converse of $A(n)$ is true for all $n \in \mathbb{N}$. (4 marks)
   (f) Use **proof by contradiction** to show that if $x, y, z \in \mathbb{Z}$ then at least one of the numbers $x+y$, $y+z$ and $x+z$ is even. [Hint: what is $(x+y)+(y+z)+(x+z)$?] (6 marks)

2. Let A and B be sets and let $f : A \to B$ be a function. Define a function $h : \mathcal{P}(B) \to \mathcal{P}(A)$ by declaring that, for $C \in \mathcal{P}(B)$, $h(C) = \{x \in A : f(x) \notin C\}$. Show that if f is a bijection then h is a bijection. (20 marks)

3. Suppose that the sequence (s_n) satifies $s_1 = 4$, $s_2 = 12$ and $s_{n+1} = 4s_n - 4s_{n-1}$ for $n \geq 2$. Use complete induction to prove that for all $n \in \mathbb{N}$, $s_n = (n+1)2^n$. (20 marks)

4. (a) Let $(X, \preceq_X)$, $(Y, \preceq_Y)$ and $(Z, \preceq_Z)$ be partially ordered sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose that g is strictly order preserving. Show that $g \circ f$ is strictly order preserving if and only if f is strictly order preserving. (12 marks)
   (b) Give an example of partially ordered sets $(X, \preceq_X)$, $(Y, \preceq_Y)$ and $(Z, \preceq_Z)$ and functions $f : X \to Y$ and $g : Y \to Z$ such that g and $g \circ f$ are both order-preserving but f is not order-preserving. (8 marks)

5. Let $a, b \in \mathbb{N}$. Prove that a and b are relatively prime if and only if for all $c \in \mathbb{N}$, if $a \mid c$ and $b \mid c$ then $ab \mid c$. [You may assume that there exist $x, y \in \mathbb{Z}$ with $\gcd(a,b) = ax + by$, and that $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$.] (20 marks)

6. (a) Let G be a group, and let H and K be subgroups of G. Show that $H \cap K$ is a subgroup of G. (10 marks)
   (b) Find subgroups H and K of S_3 such that $H \cup K$ is not a subgroup of S_3. The Cayley Table for S_3 is given below.

$*$	e	φ	ψ	α	β	γ
e	e	φ	ψ	α	β	γ
φ	φ	ψ	e	β	γ	α
ψ	ψ	e	φ	γ	α	β
α	α	γ	β	e	ψ	φ
β	β	α	γ	φ	e	ψ
γ	γ	β	α	ψ	φ	e

(10 marks)

7. Let (s_n) be a bounded increasing sequence in $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be an order-preserving function. Define a new sequence (t_n) by declaring that $t_n = f(s_n)$ for all n. Prove from first principles that (t_n) converges. (20 marks)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, b, c \in \mathbb{R}$ with $a, b > 0$. Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = af(bx)$. Suppose f is differentiable at c. Prove from first principles that g is differentiable at c, with $g'(c) = abf'(bc)$. (20 marks)

THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2004
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MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: This is an OPEN BOOK examination.
Answer ALL **EIGHT** questions. Each question carries 20 marks.

1. For any integer n, let $A(n)$ be the statement:
"If $n = 3q - 2$ or $n = 3q + 1$ for some $q \in \mathbb{Z}$, then $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$."

   (a) Write down the negation of $A(n)$. (3 marks)

   (b) Write down the contrapositive of $A(n)$. (4 marks)

   (c) Write down the converse of $A(n)$. (3 marks)

   (d) Use a **direct proof** to show that $(\forall\, n \in \mathbb{Z})\ A(n)$. (5 marks)

   (e) Use **proof by contradiction** to show that the converse of $A(n)$ is true for all $n \in \mathbb{Z}$. (5 marks)

2. Let $\sim$ be the relation defined on the set of integers $\mathbb{Z}$ by $x \sim y$ if $8 \mid (3x + 5y)$ for $x, y \in \mathbb{Z}$.

   (a) Show that $\sim$ is an equivalence relation. (10 marks)

   (b) Find all distinct equivalence classes. (10 marks)

3. (a) Use congruences to show that for any natural number $n \in \mathbb{N}$, the number $21(15n+27)(n+28)$ is divisible by 14. (7 marks)

   (b) Suppose a sequence $\{s_n\}_{n=1}^{\infty}$ satisfies $s_1 = 3, s_2 = 18$ and $s_n = 6s_{n-1} - 9s_{n-2}$ for $n \geq 3$. Use complete induction to prove that $s_n = n3^n$ for all $n \in \mathbb{N}$. (13 marks)

CONTINUED

4. Let $S = \{2, 4, 5, 6, 8, 10, 15, 18, 20\}$ and let ρ be the relation on S defined by $a\ \rho\ b$ if and only if $a \mid b$. Then (S, ρ) is a poset. [You are not asked to prove this.]

   (a) Draw a lattice diagram of (S, ρ). (5 marks)

   (b) Find all maximal and all minimal elements of S. (3 marks)

   (c) Find a subset of S which has no upper bound and no lower bound. (4 marks)

   (d) Find the greatest lower bound for $\{4, 6, 10\}$. (4 marks)

   (e) Determine whether or not the subset $\{2, 4, 20\}$ of S is well-ordered. Explain your answer to this part. (4 marks)

5. Let $A = \{x \in \mathbb{R} : x \neq 0\}$ be the set of all non-zero real numbers, and let $S = \{x \in \mathbb{Q} : x \neq 0\}$ be the set of all non-zero rational numbers and $T = \mathbb{R} \setminus \mathbb{Q}$. For any $x, y \in \mathbb{R}$ define $x * y$ by

$$x * y = 3xy,$$

where xy is the ordinary multiplication of x and y in $\mathbb{R}$.

   (a) Show that $(A, *)$ is an abelian group. (10 marks)

   (b) Show that $(S, *)$ is a subgroup of $(A, *)$. (5 marks)

   (c) Determine with reason whether or not $(T, *)$ is a subgroup of $(A, *)$. (5 marks)

6. (a) Find all integer solutions of the Diophantine equation

$$946x + 374y = 44$$

   with $0 < x < 22$. (13 marks)

   (b) Find all integers $x \in \mathbb{Z}$ such that

$$189x \equiv 28 \pmod{56}.$$

   (7 marks)

7. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and define the sequence $\{c_n\}_{n=1}^{\infty}$ by

$$c_n = \begin{cases} a_n & \text{if } b_n \leq a_n, \\ b_n & \text{if } a_n < b_n. \end{cases}$$

Suppose that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$$

for some real number $a \in \mathbb{R}$. Prove from the first principles that $\lim_{n\to\infty} c_n = a$. (20 marks)

CONTINUED

8. Let f be a function from $\mathbb{R}$ to itself defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x^3 & \text{if } 0 < x \leq 1, \\ x+2 & \text{if } 1 < x. \end{cases}$$

(a) Prove from the first principles that $f(x)$ is continuous at 0. (10 marks)

(b) Prove from the first principles that $f(x)$ is *not* continuous at 1. (10 marks)

THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2003
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: This is an OPEN BOOK examination.
Answer ALL **EIGHT** questions. All questions carry equal marks.

1. (a) Explain why $\neg((\forall x)P(x) \wedge (\exists y)Q(y))$ is logically equivalent to $(\exists x)\neg P(x) \vee (\forall y)\neg Q(y)$.

(8 marks)

(b) Prove, using any method you choose, that for any finite set A, $|A| < |\mathcal{P}(A)|$.

(12 marks)

2. (a) This question does **not** ask you to prove anything.
A theorem states that if an object x has the property $\mathcal{A}$, then it must also have the property $\mathcal{B}$.
The proof of this theorem shows that if an object x has the property $\mathcal{A}$ but not the property $\mathcal{B}$, then $x \in \emptyset$.
What kind of proof is this? Explain your answer.

(6 marks)

(b) The Fibonacci numbers F_n are defined recursively as a sequence with $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

Prove using induction that $\sum_{i=0}^{n} F_i^2 = F_n \cdot F_{n+1}$ for any $n \geq 0$.

(14 marks)

3. Let $A = \{x \in \mathbb{N} : -10 \le x \le 8\}$. Let $f : A \to A$ be defined as follows: For all $x \in A$, $f(x)$ is the remainder when x is divided by 4. [You are not asked to prove that f is a function.]

(a) (i) Find $f(7)$ and $f(-7)$. (3 marks)

(ii) Determine whether or not f is onto. (3 marks)

(b) Let $g : \mathcal{P}(A) \to \mathcal{P}(A)$ be defined as follows: For all $X \in \mathcal{P}(A)$, $g(X) = \{a \in A : f(a) \in X\}$. [You are not asked to prove that g is a function.]

(i) What is $g(\{-1, 0, 1\})$? (4 marks)

(ii) Determine whether or not g is one-to-one. (4 marks)

(c) An equivalence relation is defined on A as follows: For all $a, b \in A$, $a \sim b$ if and only if $f(a) = f(b)$. [You are not asked to prove that $\sim$ is an equivalence relation.]

(i) List all elements of the set $S = \{a \in A : a \sim 7\}$. (3 marks)

(ii) Write down all of the equivalence classes under the relation $\sim$. (3 marks)

4. Let $S = \{x \in \mathbb{N} : x \le 18$ and $\gcd(x, 5) = 1$ and $\gcd(x, 7) = 1$ and $\gcd(x, 6) \ne 1$ and $x \ne 6\}$. Let ρ be the relation on S defined by $a\rho b$ if and only if $a \mid b$. Then (S, ρ) is a poset. [You are not asked to prove this.]

(a) List the elements of S. (4 marks)

(b) Draw a lattice diagram of (S, ρ). (4 marks)

(c) Find all maximal and all minimal elements of S. (2 marks)

(d) Find a subset of S which has no upper bound. (2 marks)

(e) Find a subset of S which is bounded above but has no least upper bound. (4 marks)

(f) Determine whether or not the subset $\{2, 3, 4\}$ of S is well-ordered. Explain your answer to this part. (4 marks)

5. (a) Let $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$ be the group of all symmetries of a square, and let $S = \{R_0, R_{90}, R_{180}, R_{270}\}$ be a subset of D_4.

(i) Show that S is a subgroup of D_4. (7 marks)

(ii) Show that the group S is isomorphic to the group $(\mathbb{Z}_4, +_4)$. (7 marks)

(b) Let K and L be distinct subgroups of a finite group G such that $|K| = |L| = 2$. Using Lagrange's Theorem show that the union $K \cup L$ is not a subgroup of G. (6 marks)

CONTINUED

6. (a) Find all integer solutions of the Diophantine equation

$$135x + 40y = 15$$

with $0 < x < 17$. (10 marks)

(b) Let $\mid$ be the operation of division on the set $\mathbb{Z}_p[x]$ of polynomials over $\mathbb{Z}_p$, where p is a prime.

(i) Show that $(\mathbb{Z}_3[x], \mid)$ is not a poset. (5 marks)

(ii) Show that $(\mathbb{Z}_2[x], \mid)$ is a poset. (5 marks)

7. (a) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence such that $a_n = \dfrac{3^n}{(n+2)!}$.

(i) Show that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded. (8 marks)

(ii) Find the greatest lower bound and the least upper bound of the set $\{a_n\}_{n=1}^{\infty}$ and determine whether or not either is an element of $\{a_n\}_{n=1}^{\infty}$. Find also the limit $\lim_{n\to\infty} a_n$. (4 marks)

(b) Let A and B be two non-empty sets of real numbers which are bounded above, and let $A + B = \{a + b : a \in A, b \in B\}$. Prove

$$\sup(A + B) = \sup A + \sup B.$$

(8 marks)

8. Let $\{a_n\}_{n=0}^{\infty}$ be a convergent sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$ and let f be a function which is continuous at a. Prove from first principles that $\{f(a_n)\}_{n=0}^{\infty}$ is a Cauchy sequence. (20 marks)
