1. Let S be the set $\{1, 2, 3, 4, 5, 6\}$.

(a) Let ρ be the relation

$$\rho = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,4), (5,1), (5,5), (6,2), (6,3), (6,6)\}$$

Verify that ρ is an equivalence relation on S. Find all equivalence classes and check the collection of distinct classes is a partition of S.

Reflexive: $x \in S \implies (x, x) \in \rho$ and so $x\rho x$.
Symmetric: $(x, y) \in \rho \implies (y, x) \in \rho$.
Transitive: Suppose $(x, y) \in \rho \wedge (y, z) \in \rho$. We may suppose $x \neq y$, $x \neq z$ and $y \neq z$. Check the transitivity when $x = 1, 2, 3, 4, 5$, respectively.
Equivalence classes: $[1] = \{1, 5\}$, $[2] = \{2, 3, 6\}$ and $[4] = \{4\}$. Let $S_1 = [1]$, $S_2 = [2]$ and $S_3 = [4]$, Then $S = S_1 \cup S_2 \cup S_3$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$.

(b) Let $S_1 = \{2\}$, $S_2 = \{1, 3, 5\}$ and $S_3 = \{4, 6\}$. Verify that S_1, S_2, S_3 is a partition of S. Define an equivalence relation ρ such that each S_i is an ρ-equivalence class.

Since $S = S_1 \cup S_2 \cup S_3$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$, it follows that $\{S_1, S_2, S_3\}$ is a partition of S. Let

$$\rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,3), (3,1), (1,5), (5,1), (3,5), (5,3), (4,6), (6,4)\}.$$

Then ρ is an equivalence relation and $S_1 = [2], S_2 = [1]$ and $S_3 = [4]$.

Theorem 1 (19). *If $\sim$ is an equivalence relation on A then $\Omega = \{\, T_a : a \in A \,\}$ is a partition of A.*

Conversely, if Ω is a partition of A then the relation $\sim$ defined by declaring that

$$a \sim b \iff (\exists A' \in \Omega)(a \in A' \wedge b \in A')$$

is an equivalence relation and $\Omega = \{\, T_a : a \in A \,\}$.

$\sim$ reflexive $\Rightarrow \bigcup_{a\in A} T_a = A$.

$a\sim a \Leftrightarrow a\in T_a = \{x\in A : a\sim x\} \Rightarrow a\in \bigcup_{a\in A} T_a$.

Hence $a\sim a \ \forall a\in A \Rightarrow a\in \bigcup_{a\in A} T_a \ \forall a$ ie $\bigcup_{a\in A} T_a = A$.

Theorem 18: $\sim$ RST (equiv relation)

$$\underset{1}{a\sim b} \Leftrightarrow \underset{2}{T_a\cap T_b\neq\emptyset} \Leftrightarrow \underset{3}{T_a=T_b}$$

$$\begin{tikzcd} & 1 \arrow[dr, shift left] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \arrow[ul, shift left] \end{tikzcd}$$

1 $\Rightarrow$ 3 $\boxed{a\sim b \Rightarrow T_a=T_b}$ Let $x\in T_a$ then $(a\sim x)\wedge(a\sim b)$

$\Rightarrow (a\sim x)\wedge(b\sim a)$ [S] $((b\sim a)\wedge(a\sim x))$

$\Rightarrow (b\sim x)$ [T] $\Leftrightarrow x\in T_b$.

So $T_a\subseteq T_b$. Since by S $b\sim a$ $T_b\subseteq T_a$ and $T_a=T_b$.

3 $\Rightarrow$ 1 $\boxed{T_a=T_b \Rightarrow a\sim b}$ $T_a=T_b \Rightarrow T_a\subseteq T_b \Leftrightarrow x\in T_a \Rightarrow x\in T_b$.

but then $a\sim a$ [R] $\Rightarrow a\in T_a \Rightarrow a\in T_b \Leftrightarrow b\sim a$

$\Leftrightarrow a\sim b$ [S]

3 $\Rightarrow$ 2 $T_a=T_b \Rightarrow T_a\cap T_b\neq\emptyset$

2 $\Rightarrow$ 1 $T_a\cap T_b\neq\emptyset \Rightarrow \exists x\in T_a\cap T_b \Rightarrow \exists x: (a\sim x)\wedge(b\sim x)$

$\Rightarrow \exists x: (a\sim x)\wedge(x\sim b)$ [S]

$\Rightarrow a\sim b$.

Theorem 19: (a) $\sim$ R,S,T (equiv relation) $\Rightarrow \Omega=\{T_a\}$ is a partition

By above $\sim$ [R] $\Rightarrow \bigcup T_a = A$.

Theorem 18 shows $T_a\neq T_b \Rightarrow T_a\cap T_b=\emptyset$ $a\in T_a$ shows $\Omega\neq\emptyset$.

(b) Given Ω partition $a\sim b \Leftrightarrow (\exists A'\in\Omega)((a\in A')\wedge(b\in A'))$ is RST. (equiv. rel)

[R] $A=\bigcup_{A'\in\Omega} A' \Leftrightarrow (\forall a\in A)(a\in A'$ some $A')$

$\Rightarrow (\exists A'\in\Omega)((a\in A')\wedge(a\in A')) \Rightarrow a\sim a$.

[S] $a\sim b \Leftrightarrow (\exists A'\in\Omega)((a\in A')\wedge(b\in A'))$

$\Leftrightarrow (\exists A'\in\Omega)((b\in A')\wedge(a\in A')) \Leftrightarrow b\sim a$.

[T] $(a\sim b)\wedge(b\sim c) \Leftrightarrow (\exists A'\in\Omega)((a\in A')\wedge(b\in A'))$

$\wedge (\exists B'\in\Omega)((b\in B')\wedge(c\in B'))$

$(b\in A')\wedge(b\in B') \Rightarrow A'\cap B'\neq\emptyset \Rightarrow A'=B'$

$\Rightarrow (\exists A'\in\Omega)(a\in A' \wedge (c\in B'=A'))$