DEPARTMENT OF MATHEMATICS

MATHS 255FC | Assignment 5 solutions | Due 14th October 2004

1. [12 = 4x3] Prove each of the following specifically from the axioms given in the handout on real numbers.

(a) Given $a, b \in \mathbf{R}$, show there is a unique x such that $ax = b$.
There exists $y : ay = 1$ (multiplicative inverse). Let $x = by$ then $ax = aby = bay = b.1 = b$.
If $ax = b,\ ax' = b$ then $x - x' = 1(x - x') = ay(x - x') = ya(x - x') = yax - yax' = yb - yb = 0$, so $x = x'$.

If $x=b/a$ is defined to be $ax = b$, show:
(i) $a/b + c/d = (ad + bc)/bd$ if $b, d \neq 0$. (ii) $(a/b).(c/d) = ac/bd$ if $b, d \neq 0$.
(i) Let $a = x.b,\ c = y.d$, then $a/b + c/d = x + y$.
Now consider $z = (ad + bc)/bd = (xbd + byd)/bd = (x + y)bd/bd$, then $z(bd) = (x + y)bd$ so $z = x + y$.
(ii) $z = ac/bd = xbyd/bd = xybd/bd$, so $zbd = xybd$, and $z = xy$.

(b) (i) $x < y,\ y < z \Rightarrow x < z$ (ii) $x < y,\ z > 0 \Rightarrow xz < yz$.
(i) $x < y,\ y < z \Leftrightarrow y - x,\ z - y \in P \Rightarrow z - y + y - x = z - x \in P \Leftrightarrow x < z$.
(ii) $x < y,\ z > 0 \Leftrightarrow y - x,\ z \in P \Rightarrow z(y - x) = zy - zx \in P \Rightarrow xz < yz$.

2. [12=3x4] (a) Show that (i) $|x + y| \leq |x| + |y|$ (ii) $|xy| = |x|.|y|$
(i) Show that $|x + y| \leq |x| + |y|$.
$x, y \geq 0$ or $x, y \leq 0$ $|x + y| = |x| + |y|$,
$x \geq 0, y < 0$ $|x + y| = |x| - |y| < |x| + |y|$ $\quad x < 0, y \geq 0$ $|x + y| = -|x| + |y| < |x| + |y|$.
(ii) $x.y \geq 0,\ |xy| = x.y = |x|.|y| = |-x|.|-y|,\ x.y < 0,\ |xy| = -x.y = |x|.|-y| = |-x|.|y|$.

(b) Use (a) to prove by induction that $|x_1 + x_2 + \ \dots\ + x_n| \leq |x_1| + |x_2| + \ \dots\ + |x_n|$.

True for n = 2 by (a) So if $|x_1 + x_2 + \ \dots\ + x_k| \leq |x_1| + |x_2| + \ \dots\ + |x_k|$
$|x_1 + x_2 + \ \dots\ + x_{k+1}| = |x_1 + x_2 + \ \dots\ + x_k + x_{k+1}| \leq |x_1 + x_2 + \ \dots\ + x_k| + |x_{k+1}| \leq |x_1| + |x_2| + \ \dots\ + |x_k| + |x_{k+1}|$

(c) Show that the distance function $d(x, y) = |x - y|$ obeys the triangle law: $d(x, z) \leq d(x, y) + d(y, z)$.
$d(x, z) = |z - x| = |z - y + y - x| \leq |z - y| + |y - x| = d(x, y) + d(y, z)$

3. [7] $A, B \subseteq \mathbf{R},\ A, B \neq \varnothing,\ A \subseteq B$ and B is bounded above, show lubA $\leq$ lubB.

$\forall a \in A, a \in B,\ \forall b \in B,\ b \leq \text{lub}\, B \Rightarrow \forall a \in A,\ a \leq \text{lub}\, B$ so lubB an upper bound for A i.e. lubA $\leq$ lubB.

4. [12=6x2] Find the least upper bound and greatest lower bound of each of the following subsets of **R** if they exist
and determine if either of these is an element of the set concerned.
(i) $\varnothing$ (ii) $[0, \infty)$ (iii) $[-\pi, \pi)$ (iv) $[-\pi, \pi) \cap \mathbf{Q}$ (v) $\left\{ \frac{1}{\sqrt{n}} : n \in \mathbf{N} \right\}$ (vi) $\left\{ \left(1 + \frac{2}{n}\right)^n n = 1, 2,\ \dots \right\}$

(i) No least upper bound exists. Any real x is an upper bound for $\varnothing$ since there is no $\varphi \in \varnothing,\ \varphi > x$. Similarly no glb exists. (ii) No lub unbounded above. glb = $0 \in [0, \infty)$. (iii) glb = $-\pi$ in S lub = π not in S (iv) the same except that now $-\pi$ is not in S as it is not in **Q**. (v) glb = 0 not in S lub = 1 in S (vi) The sequence is monotone increasing (limit of compound interest) the first term is 3 (glb) and the limit is e^2 (lub).

5. [12=2x6] (a) Show first principles that the sequence $\left\{\frac{4n+1}{n+4},\ n=0,\ 1,\ 2,\ \ldots\right\}$ is convergent.

$$\left|\frac{2n+1}{n+2}-2\right|<\varepsilon \Leftrightarrow \left|\frac{2n+1-4-2n-4}{n+2}\right|<\varepsilon \Leftrightarrow \left|\frac{-3}{n+2}\right|<\varepsilon \Leftrightarrow 3<\varepsilon(n+2) \Leftrightarrow 3<\varepsilon n+2\varepsilon \Leftrightarrow 3-2\varepsilon<\varepsilon n$$

$$\Leftrightarrow \frac{3-2\varepsilon}{\varepsilon}<n.\ \square\text{Hence choose } N(\varepsilon)=\frac{3-2\varepsilon}{\varepsilon} \text{ then } \forall\varepsilon>0, \exists N(\varepsilon)=\frac{3-2\varepsilon}{\varepsilon}>0 : n>N \Rightarrow \left|\frac{2n+1}{n+2}-2\right|<\varepsilon.$$

(b) Hence or otherwise show that the sequence $\left\{\sqrt{\frac{2n+1}{n+2}}, n=0,1,2,\ldots\right\}$ is convergent.

Note that $\frac{2(n+1)+1}{(n+1)+2}-\frac{2n+1}{n+2}=\frac{3}{(n+3)(n+2)}>0$ so $\frac{2n+1}{n+2}$ is monotone increasing.

Also $2-\frac{2n+1}{n+2}=\frac{3}{n+2}>0$ so $\frac{2n+1}{n+2}$ is bounded above by 2.

Since $\sqrt{x}$ is a monotone increasing function , $x<y \Rightarrow \sqrt{x}<\sqrt{y}$ so $\sqrt{\frac{2n+1}{n+2}}$ is monotone increasing, and bounded above by $\sqrt{2}$. Hence it is convergent.

6. [15=5x3] For each of the following sequences, determine whether or not it is:

(a) convergent and if so find its limit,
(b) bounded and if so find a convergent subsequence
(c) find a subsequence which is increasing. or one which is decreasing, or both if possible.

(i) $\left\{1^n+(-1)^n:\ n\in\mathbf{N}\right\}$ (ii) $\left\{\frac{1}{n^3}:n\in\mathbf{N}\right\}$ (iii) $\left\{\frac{1^n}{n^3}+\frac{(-1)^n}{n^3}:n\in\mathbf{N}\right\}$ (iv) $\left\{n^n:n\in\mathbf{N}\right\}$ (v) $\left\{\frac{n!}{n^n}:\ n\in\mathbf{N}\right\}$

(i) Not convergent as alternating between 2 and 0. Bounded. Absolute value bounded by 2. A convergent subsequence 2, 2, 2, 2 .... or 0, 0, 0, 0 ... hese are constant so are both increasing and decreasing.

(ii) $a_{n+1}-a_n=\frac{1}{(n+1)^3}-\frac{1}{n^3}=-\frac{3n^2+3n+1}{n^3(n+1)^3}<0$, so monotone decreasing and bounded below. Hence convergent. The limit is zero $|a_n|=\left|\frac{1}{n^3}\right|<\varepsilon$ if $n>\varepsilon^{-\frac{1}{3}}$. Bounded, decreasing and a;rady convergent.

(iii) This is basically a product of (i) and (ii) It is a product of a sequence converging to zero and a bounded sequence so it is bounded and convergent to 0. The odd terms form a decreasing subsequence $\left\{2,\frac{2}{27},\frac{2}{125},\ldots\right\}$.

(iv) Increasing unbounded, not convergent.

(v) $a_{n+1}-a_n=(n+1)!n^n-n!(n+1)^{n+1}=n!(n+1)\left(n^n-(n+1)^n\right)<0$ so a_n decreasing bounded below by zero and hence convergent. Since $\frac{a_{n+1}}{a_n}=\frac{(n+1)!n^n}{n!(n+1)^{n+1}}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}\to\frac{1}{e}<1$ the terms must get arbitrarily small, so the limit is zero. If we want to we can check this by showing there is an N so that for n > N $a_n<a_Nr^n, \square<1$.

7. [10=2x5] If $\{x_n\}$ is bounded, consider the sequence $\{g_n=\text{glb}\{x_k:k\geq n\}\}$

(a) Show $\{g_n\}$ is bounded.

Note that $\{g_n\}$ is monotone increasing since for $n>m$ $\{x_k:k\geq n\}\subseteq\{x_k:k\geq m\}$ so

$g_n = \text{glb}\{x_k : k \geq n\} \geq \text{glb}\{x_k : k \geq m\} = g_m$.

If $\{g_n\}$ unbounded, $\forall y \in \mathbf{R}\ \exists g_n = \text{glb}\{x_n, x_{n+1}, \ \ldots\ \} > y$, so $\exists x_p\ \ p \geq n : x_p > y$ so $\{x_n\}$ unbounded.

(b) Show that if $\{x_n\}$ is convergent then $\{g_n\}$ is also and $\{x_n\}$ and $\{g_n\}$ have the same limit.

(b) If $x_n \to l$ as $n \to \infty$, show $g_n \to l$ as $n \to \infty$.

If $\{g_n\} \not\to l$ then $\exists d > 0 : |g_n - l| > d\ \ \forall n > N$. As noted $\{g_n\}$ is monotone increasing, so $g_n \leq l$. Hence by the defn $g_n = \text{glb}\{x_n, x_{n+1}, \ \ldots\ \}\ \forall \varepsilon > 0\ \ \exists x_p\ \ p \geq n : x_p < g_n + \varepsilon$, so $|x_p - l| > d - \varepsilon$. So $\{x_n\} \not\to l$.