(**Total = 97 marks**)

1. (**8 marks**) Show $12 \mid n^4 - n^2$. P(1) true $1^4 - 1^2 = 0.12 \mid 0$. If P(k) then $k^4 - k^2 = 12p$. So $k^4 = 12p + k^2$, then $(k+1)^4 - (k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - (k^2 + 2k + 1)$

$$\begin{aligned}(k+1)^4 - (k+1)^2 &= k^4 + 4k^3 + 6k^2 + 4k + 1 - (k^2 + 2k + 1)\\ &= k^4 + 4k^3 + 5k^2 + 2k\\ &= 12p + k^2 + 4k^3 + 5k^2 + 2k\\ &= 12p + 2k(k^2 + 4k^2 + 6k + 2)\\ &= 12p + 2k(k+1)(2k+3)\end{aligned}$$

Now one of k and $k+1$ are even so there is an additional factor of 2 in $2k(k+1)(2k+3)$. In addition if neither k nor $k+1$ are factors of 3 then $k = 3q+1$ (check) and so $2k+1 = k+(k+1) = 3q+1+3q+2 = 6q+3$ is a multiple of 3. Hence $12 \mid 2k(k+1)(2k+3)$ and P(k+1) is true. Hence P(n) is true for all $n \in \mathbb{N}$.

2. (**7 marks**) Show for $n \geq 10$ that $2^n > n^3$. P(10) true , i.e. $2^{10} = 1024 > 1000 = 10^3$.
If P(k) true $2^k > k^3$ then $2^{k+1} = 2.2^k > k^3 + k^3 > k^3 + 10k^2$ (Since $k \geq 10$) $> k^3 + 3k^2 + 3k^3 + 3k^2 > k^3 + 3k^2 + 3k + 1 = (k+1)^3$. Hence P(k+1) is true, and so P(n) is true for all $n \geq 10$.

3. (**7 marks**) Use induction to show that $\sum_{i=1}^{n}(2n-1)^2 = 1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n}{3}(2n+1)(2n-1)$. P(1) true $1^2 = 1 = \frac{1}{3}(2+1)(2-1)$. If P(k) i.e. $1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{k}{3}(2k+1)(2k-1)$ then $\sum_{i=1}^{k+1}(2k-1)^2 = 1^2 + 3^2 + \cdots + (2k-1)^2 + (2(k+1)-1)^2 = \frac{k}{3}(2k+1)(2k-1) + (2k+1)^2 = (2k+1)(\frac{k(2k-1)}{3} + 2k+1) = (2k+1)\frac{2k^2+5k+3}{3} = \frac{k+1}{3}(2k+1)(2k+3) = \frac{k+1}{3}(2((k+1)+1)(2(k+1)-1)$. Since this is precisely the sum formula for $k+1$, P(k+1) is true and hence P(n) is true for all $n \in \mathbb{N}$.

4. (**18=5+5+4+4 marks**) A function $f : \mathbb{Q} \to \mathbb{Q}$ is a *flat* function if for all $m, n \in \mathbb{Q}$, $f(m+n) = f(m) + f(n)$. Suppose f is a flat function and $f(1) = a$. Show using induction that:

(a) $f(kn) = kf(n)$ for all $k \in \mathbb{N}, n \in \mathbb{Q}$. P(1) true since $f(1.n) = f(n) = 1.f(n)$. If P(k) i.e. $f(kn) = kf(n)$ then $f((k+1)n) = f(kn+n) = f(kn) + f(n) = kf(n) + f(n) = (k+1)f(n)$, so P(k+1) true. Hence P(n) true for all $n \in \mathbb{N}$.

(b) $f(n) = an$ for all $n \in \mathbb{N}$. P(1) f(1)=a true. If P(k) then $f(k+1) = f(k) + f(1) = ka + a = (k+1)a$, so P(k+1) true. Hence P(n) true for all $n \in \mathbb{N}$.

(c) $f(n) = an$ for all $n \in \mathbb{Z}$. $f(n) = f(n+0) = f(n) + f(0)$, hence $f(0) = 0 = 0.n$. Now $f(-n) + f(n) = f(-n+n) = f(0) = 0$, so $f(-n) = -f(n) = (-n)a$.

(d) $f(n) = an$ for all $n \in \mathbb{Q}$. From (a) $ma = f(m) = f(n\frac{m}{n}) = nf(\frac{m}{n})$, so $f(\frac{m}{n}) = \frac{m}{n}a$. Now $f(\frac{-m}{n}) + f(\frac{m}{n}) = f(0) = 0$, so $f(\frac{-m}{n}) = \frac{m}{n}a$.

5. (**12=4x3 marks**)

(a) Note that $\rho = \{(x,y) \in A \times A : xy = 0\} = \emptyset$ since both x and y are positive. So ρ is **symmetric, antisymmetric** and **transitive**. But ρ is **not reflexive** because $(1,1) \notin \rho$.

(b) Note $\rho = \{(0,4),(4,0),(1,3),(3,1),(2,2)\}$. So ρ is **not reflexive** because $(0,0) \notin \rho$.
Symmetric: $(x,y) \in \rho \iff x+y=4 \iff y+x=4 \iff (y,x) \in \rho$;
Not antisymmetric: $0\rho 4 \wedge 4\rho 0$ but $4 \neq 0$.
Not transitive: $0\rho 4 \wedge 4\rho 0$ but $(0,0) \notin \rho$.

(c) **Not reflexive**: $(4,4) \notin \rho$.
Not symmetric: $(2,1) \in \rho$ but $(1,2) \notin \rho$.
Not antisymmetric: $2\rho 4 \wedge 4\rho 2$ but $4 \neq 2$.
Not Transitive: $4\rho 2 \wedge 2\rho 1$ but $(4,1) \notin \rho$.

(d) **Reflexive**: for all $x \in D$, $x - x = 0 \in \mathbb{Z}$.
Symmetric: $(x,y) \in \rho \iff x-y \in \mathbb{Z} \iff y-x = -(x-y) \in \mathbb{Z} \iff (y,x) \in \rho$.
Not antisymmetric: $1\rho 2 \wedge 2\rho 1$ but $1 \neq 2$.
Transitive: $x\rho y \wedge y\rho z \iff x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$, so $x-z = (x-y)+(y-z) \in \mathbb{Z}$ and $x\rho y$.

6. (**10=6+4 marks**) To show that $\preceq$ is a partial order we must show that it is reflexive, antisymmetric and transitive.

Reflexive: Let $x \in \mathbb{N}$. Then $x = x$ so $x \preceq x$.

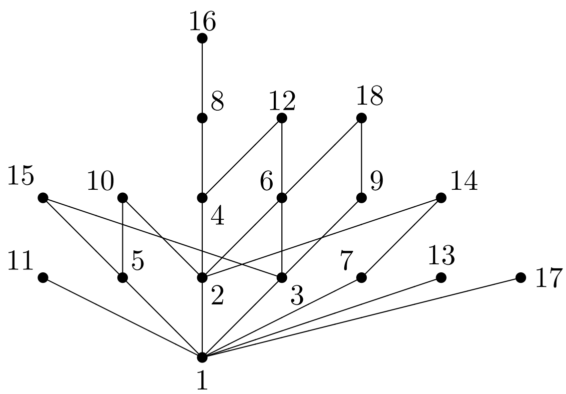
Antisymmetric: Let $x, y \in \mathbb{N}$ with $x \preceq y$ and $y \preceq x$. Suppose, for a contradiction, that $x \neq y$. Then we must have $2x \leq y$ and $2y \leq x$. We have $x \leq 2x \leq y \leq 2y \leq x$, so $x = y$, a contradiction. So we must have $x = y$.

Transitive: Let $x, y \in \mathbb{N}$ with $x \preceq y$ and $y \preceq z$. If $x = y$ then (since $y \preceq z$) we have $x \preceq z$, and similarly if $y = z$ then $x \preceq y = z$ so $x \preceq z$. So suppose that $2x \leq y$ and $2y \leq z$. Since $y \leq 2y$ we have $2x \leq y \leq 2y \leq z$, so $2x \leq 4x \leq z$, so $x \preceq z$, as required.

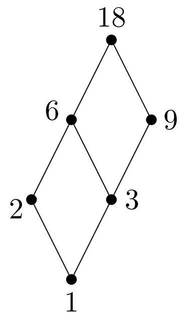
To show that $\preceq$ is not a total order, we exhibit a counterexample: we have $2 \neq 3$ and $2.2 \not\leq 3$ and $2.3 \not\leq 2$ so $2 \not\preceq 3$ and $3 \not\preceq 2$.

7. (**16=4x4 marks**)

(a) We have the lattice diagrams



and



(b) 1 minimal and a least element 10,11, 15, 13, 17, 14, 18, 12, 16 all maximal none greatest. 18 greatest and maximal and 1 least and minimal in the second lattice.

(c) The least upper bound for $\{1,2,3\}$ in B is 6.

(d) There are a number of choices. One would be $\{2,3,5\}$: any upper bound for this set would have to be divisible by 2, 3 and 5, so would have to be at least 30. Another choice would be to take any two maximal elements, say $\{12,18\}$.

8. (**10=5+5 marks**) Since S has a lower bound, b_0 say, we have $b_0 \in L_S$ so $L_S \neq \emptyset$. Since $S \neq \emptyset$, there is some $s_0 \in S$. Now, for every $b \in L_S$ we have $b \preceq s$ for all $s \in S$, and in particular $b \preceq s_0$. So s_0 is an upper bound for L_S, so L_S has at least one upper bound.

Let $g = \sup L_S$. We must show that g is a lower bound for S, i.e. that $g \preceq s$ for all $s \in S$. So let $s \in S$. As above, for any $b \in B$ we must have $b \preceq s$. Thus s is an upper bound for L_S. Since g is the **least** upper bound for L_S, we must have $g \preceq s$. Since this holds for all $s \in S$, g is a lower bound for S, as required.

Finally, we must show that g is a **greatest** lower bound. So let b be a lower bound for S. Then $b \in L_S$, so (since g is an upper bound for L_S) $b \preceq g$, as required.

9. (**9=5+4 marks**) We must show that ρ is reflexive, antisymmetric and transitive.

Reflexive: Let $(x,y) \in \mathbb{R}^2$. Then $\mid x \mid + \mid y \mid = \mid x \mid + \mid y \mid$, so $(x,y)\ \rho\ (x,y)$.

Symmetric: Let $(x,y),(u,v) \in \mathbb{R}^2$ with $(x,y)\ \rho\ (u,v)$. Then $\mid x \mid + \mid y \mid = \mid u \mid + \mid v \mid$, so $\mid u \mid + \mid v \mid = \mid x \mid + \mid y \mid$, so $(u,v)\ \rho\ (x,y)$.

Transitive: Let $(x,y),(u,v),(z,w) \in \mathbb{R}^2$ with $(x,y)\ \rho\ (u,v)$ and $(u,v)\ \rho\ (z,w)$. Then $\mid x \mid + \mid y \mid = \mid u \mid + \mid v \mid$ and $\mid u \mid + \mid v \mid = \mid z \mid + \mid w \mid$, so $\mid x \mid + \mid y \mid = \mid z \mid + \mid w \mid$, i.e. $(x,y)\ \rho\ (z,w)$.

Notice that $(x,y)\ \rho\ (u,v)$ iff (x,y) and (u,v) are on the same straight line segment i.e. $x+y = u+v = c$ in the first quadrant, or a reflection of such points in either of the axes (to allow for $\mid x \mid + \mid y \mid = c$). Thus $T_{(x,y)}$ is the square diamond centred at the origin of diameter $2c$. In the special case where $(x,y) = (0,0)$ we have $T_{(0,0)} = \{(0,0)\}$.