(**Total 80 marks**)

1. (**15=2+3+2+2+3+3 marks**)

(a) "19 is a prime number" is a statement, which we would translate as $P(19)$ (where $P(n)$ denotes "n is prime").

(b) "If n is even then n is not prime" is a predicate, which we would translate as $E(n) \implies \sim P(n)$ (where $E(n)$ denotes "n is even and $P(n)$ denotes "n is prime"). If we assume the predicate for all such n it becomes a statement we can prove false by counterexample. If n is even then $n = 2.p$ so n has factors other than itself and 1 so is not prime, except when $n = 2$, when n is both prime and even. Counterexample.

(c) "Is 13 a prime number?" is neither a statement nor a predicate. It is a question.

(d) "Solve $x^2 - 4 = 0$" is neither a statement nor a predicate. It is a command.

(e) "Every even number is the sum of two odd numbers" is a statement, which we would symbolise as $(\forall n \in \mathbb{N})(E(n) \implies (\exists m, k)(O(m) \wedge O(k) \wedge S(m, k, n)))$ (where $E(n)$ denotes "n is even" and $S(m, k, n)$ denotes "$m + k = n$"). It is true since if n is even then $n = 2.p = p + p = (p - 1) + (p + 1)$. If p is odd then the first expression gives two odd numbers and if p is even the second expression dies likewise.

(f) "There exists a real number n such that for all real r, $r.n = n.r = n$" is a statement, which we would symbolise as $(\exists n \in \mathbb{R})(\forall r \in \mathbb{R})(n.r = r.n = n)$. The statement is true since $0.r = r.0 = 0\,\forall r$

2. (**12=4x3 marks**)

(a) We have the truth table

A	B	$(\sim$	$A)$	$\vee$	B	$\wedge$	$\sim$	$(A$	$\implies$	$B)$
T	T	F	T	T	T	F	F	T	T	T
T	F	F	T	F	F	F	T	T	F	F
F	T	T	F	T	T	F	F	F	T	T
F	F	T	F	T	F	F	F	F	T	F

From this we see that the main column contains only Fs, so the statement is a contradiction.

(b) We have the truth table

A	B	$\sim$	$(A$	$\wedge$	$B)$	$\implies$	$\sim$	$(A$	$\vee$	$B)$
T	T	F	T	T	T	T	F	T	T	T
T	F	T	T	F	F	F	F	T	T	F
F	T	T	F	F	T	F	F	F	T	T
F	F	T	F	F	F	T	T	F	F	F

Since both Ts and Fs occur in the principal column, this is neither a tautology nor a contradiction.

(c) We have the truth table

A	B	$(A$	$\Longrightarrow$	$B)$	$\Longleftrightarrow$	$(A$	$\vee$	$\sim$	$B)$
T	T	T	T	T	T	T	T	F	T
T	F	T	F	F	F	T	T	T	F
F	T	F	T	T	F	F	F	F	T
F	F	F	T	F	T	F	T	T	F

Again this is neither.

(d) We have the truth table

A	B	$(\sim$	A	$\Longrightarrow$	$B)$	$\Longrightarrow$	$(\sim$	B	$\Longrightarrow$	$A)$
T	T	F	T	T	T	T	F	T	T	T
T	F	F	T	T	F	T	T	T	T	T
F	T	T	F	T	T	T	F	F	T	F
F	F	T	F	F	F	T	T	F	F	F

This statement is a tautology.

3. (**12=1+1+1+1+2+2+2+2 marks**)

(a) The contrapositive of $A(n)$ is "If n^2+1 is even then n is not a multiple of 4".

(b) The converse of $A(n)$ is "If n^2+1 is odd then n is a multiple of 4".

(c) The negation of $A(n)$ is "n is a multiple of 4 and n^2+1 is even".

(d) $A(n)$ is true for some $n \in \mathbb{N}$: for example, $A(4)$ is true (since 4 is even and $17 = 4^2+1$ is odd).

(e) $A(n)$ is true for all $n \in \mathbb{N}$. We give a direct proof. Suppose n is a multiple of 4. Then $n = 4k$ for some $k \in \mathbb{Z}$, and so $n^2+1 = 16k^2+1 = 2(8k^2)+1$, and $8k^2 \in \mathbb{Z}$, so n^2+1 is odd.

(f) By (d) and (e), the contrapositive is true for some $n \in \mathbb{N}$, and indeed for all $n \in \mathbb{N}$, since it is equivalent to $A(n)$ itself.

(g) The converse of $A(n)$ is true for some $n \in \mathbb{N}$: for example the converse of $A(4)$ is (vacuously) true since $4^2+1 = 3$ is odd and 4 is a multiple of 4.

(h) The converse of $A(n)$ is not true for all $n \in \mathbb{N}$. We give a counterexample. $2^2+1 = 5$ odd but 2 is not a multiple of 4.

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4. (**8 marks**) If $\sqrt[3]{3}$ is rational then it can be expressed as $3 = \frac{p^3}{q^3}$ with p and q having no common factors. So $p^3 = 3q^3$. So p^3 is a multiple of 3. So p is a multiple of 3. So $p = 3k$. So $p^3 = 27k^3 = 3q^3$. I.e. $3(k^3) = q^3$. So q^3 is a multiple of 3. So q is a multiple of 3. Contradicting p, q having no common factors.

5. (**9=3x3 marks**)

(a) Suppose $m < n$. Then (since $m, n \geq 0$) $m^3 < mmn < mnn < n^3$, so $m^3 + m < n^3 + n$, i.e. $f(m) < f(n)$.

(b) Suppose $m \not< n$. Then $n \leq m$. If $n < m$ then, by (a), $f(m) < f(n)$, and if $m = n$ then $f(m) = f(n)$. So we have $f(n) \leq f(m)$, so $f(m) \not< f(n)$. Hence, by contraposition, if $f(m) < f(n)$ then $m < n$.

(c) Let $m, n \in \mathbb{N}$. [We must show that if $f(m) = f(n)$ then $m = n$.] Suppose, for a contradiction, that $f(m) = f(n)$ but $m \neq n$. Since $m \neq n$ we have $m < n$ or $n < m$. So, by (a), we have $f(m) < f(n)$ or $f(n) < f(m)$. Either way, we have $f(m) \neq f(n)$, contradicting our assumption that $f(m) = f(n)$. Hence if $f(m) = f(n)$ then $m = n$, in other words f is one-to-one.

6. (a) (**3 marks**) Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $A \setminus B = \{1\}$, but $A \cup B = \{1, 2, 3\}$ and $A \cap B = \{2\}$. Thus $(A \cup B) \setminus (A \cap B) = \{1, 3\}$.

(b) (**3 marks**) $B \subseteq C \iff (x \in B \implies x \in C)$.
$x \in B \setminus A \implies (x \in B) \wedge (x \notin A) \implies (x \in C) \wedge (x \notin A) \implies x \in C \setminus A$.

(c) (**3 marks**) $A \subseteq B \iff (x \in A \implies x \in B)$.
So $A \wedge B^{\mathcal{C}} \neq \varnothing \iff \exists x : (x \in A) \wedge (x \in B^{\mathcal{C}}) \implies x \in B \wedge (x \in B^{\mathcal{C}})$. Contradiction.
On the other hand $A \wedge B^{\mathcal{C}} = \varnothing \implies ((x \in A) \implies (x \notin B^{\mathcal{C}}) \implies x \in B))$, so $A \subseteq B$.

(d) (**3 marks**) $(B \setminus A) \cup (A \cap B) = (B \cap A^{\mathcal{C}}) \cup (B \cap A) = A$.
$B \subseteq A \cup B$, so $A \cup B = B \iff A \cup B \subseteq B \implies A \subseteq B$ since
$((x \in A) \vee (x \in B) \implies (x \in B)) \implies ((x \in A) \implies (x \in B))$.

(e) (**5 marks**)

$$
\begin{aligned}
x \in A \setminus \bigcup_{\alpha \in \Lambda} B_\alpha &\iff x \in A \wedge x \notin \bigcup_{\alpha \in \Lambda} B_\alpha \\
&\iff x \in A \wedge \sim\!(x \in \bigcup_{\alpha \in \Lambda} B_\alpha) \\
&\iff x \in A \wedge \sim\!(\exists \alpha \in \Lambda)(x \in B_\alpha) \\
&\iff x \in A \wedge (\forall \alpha \in \Lambda)(x \notin B_\alpha) \\
&\iff (\forall \alpha \in \Lambda)(x \in A \wedge x \notin B_\alpha) \\
&\iff (\forall \alpha \in \Lambda)(x \in A \setminus B_\alpha) \\
&\iff x \in \bigcap_{\alpha \in \Lambda} (A \setminus B_\alpha)
\end{aligned}
$$

Thus $A \setminus \bigcap_{\alpha \in \Lambda} B_\alpha = \bigcup_{\alpha \in \Lambda} (A \setminus B_\alpha)$.

7. (**10=2+3+2+3 marks**)

(a) $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}, \{1,2,4\}, \{1,2,5\}, \{1,4,5\}, \{2,4,5\}, \{1,2,4,5\}\}$. So $\|\mathcal{P}(A)\| = 2^4 = 16$.

(b) $\mathcal{P}(B) = \{\varnothing, \{3\}, \{5\}, \{6\}, \{3,5\}, \{3,6\}, \{5,6\}, \{3,5,6\}\}$.

(c) $\mathcal{P}(A \cap B) = \mathcal{P}(\{5\}) = \{\varnothing, \{5\}\}$.

(d) $\|\mathcal{P}(A \cup B)\| = \|\mathcal{P}(\{1,2,3,4,5,6\})\| = 2^6 = 64$.