DEPARTMENT OF MATHEMATICS

MATHS 255FC — Assignment 5 — Due 14th October 2004

1. Prove each of the following specifically from the axioms of the real numbers.

(a) Given $a, b \in \mathbf{R}$, $a \neq 0$, show there is a unique x such that $ax = b$.
If $x=b/a$ is defined to be $ax = b$, show
(i) $a/b + c/d = (ad + bc)/bd$ if $b, d \neq 0$. (ii) $(a/b).(c/d) = ac/bd$ if $b, d \neq 0$.

(b) (i) $x < y,\ y < z \Rightarrow x < z$ (ii) $x < y,\ z > 0 \Rightarrow xz < yz$.

2. If F is an ordered field we define $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

(a) Show that (i) $|x + y| \leq |x| + |y|$ (ii) $|xy| = |x|.|y|$

(b) Use (ai) to prove by induction that $|x_1 + x_2 + \ \ldots \ + x_n| \leq |x_1| + |x_2| + \ \ldots \ + |x_n|$.

(c) Show that the distance function $d(x, y) = |x - y|$ obeys the triangle law: $d(x, z) \leq d(x, y) + d(y, z)$.

3. If $A, B \subseteq \mathbf{R}$, $A, B \neq \varnothing$, $A \subseteq B$ and B is bounded above, show $\text{lubA} \leq \text{lubB}$.

4. Find the least upper bound and greatest lower bound of each of the following subsets of **R** if they exist and determine if either of these is an element of the set concerned.
(i) $\varnothing$ (ii) $[0, \infty)$ (iii) $[-\pi, \pi)$ (iv) $[-\pi, \pi) \cap \mathbf{Q}$ (v) $\left\{\frac{1}{\sqrt{n}} : n \in \mathbf{N}\right\}$ (vi) $\left\{\left(1 + \frac{2}{n}\right)^n n = 1, 2, \ \ldots\right\}$

5. (a) Show first principles that the sequence $\left\{\frac{2n+1}{n+2}, n = 0, 1, 2, \ldots\right\}$ is convergent.

(b) Hence or otherwise show that the sequence $\left\{\sqrt{\frac{2n+1}{n+2}}, n = 0, 1, 2, \ldots\right\}$ is convergent.

6. For each of the following sequences, determine whether or not it is:

(a) convergent and if so find its limit,
(b) bounded and if so find a convergent subsequence
(c) find a subsequence which is increasing. or one which is decreasing, or both if possible.

(i) $\{1^n + (-1)^n :\ n \in \mathbf{N}\}$ (ii) $\left\{\frac{1}{n^3} : n \in \mathbf{N}\right\}$ (iii) $\left\{\frac{1^n}{n^3} + \frac{(-1)^n}{n^3} : n \in \mathbf{N}\right\}$ (iv) $\{n^n : n \in \mathbf{N}\}$ (v) $\left\{\frac{n!}{n^n} :\ n \in \mathbf{N}\right\}$

7. If $\{x_n\}$ is bounded, consider the sequence $\{g_n = \text{glb}\{x_k : k \geq n\}\}$

(a) Show $\{g_n\}$ is bounded.
(b) Show that if $\{x_n\}$ is convergent then $\{g_n\}$ is also and $\{x_n\}$ and $\{g_n\}$ have the same limit.