NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

PLEASE SHOW ALL WORKING.

1. Let $A = \{2 - \frac{1}{2^n} : n \in \mathbb{N}\}$ and view A as a totally ordered set under the usual ordering on $\mathbb{R}$. Show $A \simeq \mathbb{N}$ as posets by showing the obvious function from $\mathbb{N}$ to A is an order isomorphism.

2. Let $f : A \to B$ be a function. Define a new function $F : \mathcal{P}(A) \to \mathcal{P}(B)$ by declaring that, for $S \subseteq A$,

$$F(S) = \{\, f(a) : a \in S \,\}.$$

Show that F is one-to-one if and only if f is one-to-one.

3. Let $f : A \to B$ and $g : B \to C$ be functions. Show that if $g \circ f$ is one-to-one and f is onto then g is one-to-one.

4. Let $f : X \to Y$ be a function, let Λ be a nonempty indexing set, and for each $\alpha \in \Lambda$ let S_α, A, B be a subsets of Y.

(a) Show that $f^{-1}(S^{\complement}_Y) = f^{-1}(S)^{\complement}_X$.

(b) Show that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) Show that $f^{-1}(A \setminus \bigcap_{\alpha\in\Lambda} S_\alpha) = \bigcup_{\alpha\in\Lambda} f^{-1}(A \setminus S_\alpha)$.

(d) Show that if $A, B \subseteq X$, $f(A \setminus B) \supseteq f(A) \setminus f(B)$.

(e) Find a function $f : S \to S$ and subsets $A, B \subseteq S$ such that $f(A \setminus B) \neq f(A) \setminus f(B)$.

5. Show that given the axioms for $\mathbb{N}$ and $\mathbb{Z}$, there exists a unique $x : a + x = b$ and hence that $b - c$ can be defined for all $b, c \in \mathbb{Z}$ according to the definition in the notes. Use this to show that

(a) for all $x \in \mathbb{Z}$, $-(-x) = x$; and

(b) for any $x \in \mathbb{Z}$, $-x = (-1) \cdot x$.

(c) for all $x, y, z \in \mathbb{Z}, x.(y - z) = x.y - x.z$.

6. Use the modified version of Euclid's Algorithm to find $\gcd(a, b)$ and integers x and y with $\gcd(a, b) = ax + by$ for the following pairs of integers.

(a) 72 and 78.

(b) 928 and 2944.

(c) 1173 and 957.

7. Show that if $a, b, c, x, y \in \mathbb{Z}$ with $c \mid a$ and $c \mid b$ then $c \mid ax + by$.

8. Recall that if $r, s \in \mathbb{Z}$, we say that r and s are *relatively prime* if the only positive common divisor of r and s is 1, in other words if $\gcd(r, s) = 1$.

(a) Show that if there exist $x, y \in \mathbb{Z}$ with $rx + sy = 1$ then r and s are relatively prime.

(b) Show that if r and s are relatively prime and $r \neq s$, then so are s and $r - s$.

(c) Let $a, b \in \mathbb{N}$ and suppose $a = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t}$, where $p_1, \ldots, p_t$ are distinct primes and α_i, β_i are nonnegative integers.
Let $m_1 = \min\{\alpha_i, \beta_i\}$. Show that $\gcd(a, b) = p_1^{m_1} p_2^{m_2} \ldots p_t^{m_t}$

(d) Suppose $m_1, m_2, \ldots, m_k$ are integers and p is a prime number. Use induction to show that if $p \mid m_1 m_2 \cdots m_k$ then $p \mid m_i$ for some i.