NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. Show by induction that for all $n \in \mathbb{N}$, $12 \mid n^4 - n^2$.
[You will need to check some cases for k during the process.]

2. Show that for all $n \geq 10$, $2^n > n^3$.

3. Use induction to show that $\sum_{i=1}^{n}(2n-1)^2 = 1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n}{3}(2n+1)(2n-1)$.

4. A function $f : \mathbb{Q} \to \mathbb{Q}$ is a *flat* function if for all $m, n \in \mathbb{Q}$, $f(m+n) = f(m) + f(n)$. Suppose f is a flat function and $f(1) = a$. Show using induction that:

(a) $f(kn) = kf(n)$ for all $k \in \mathbb{N}, n \in \mathbb{Q}$.
(b) $f(n) = an$ for all $n \in \mathbb{N}$.
(c) $f(n) = an$ for all $n \in \mathbb{Z}$.
(d) $f(n) = an$ for all $n \in \mathbb{Q}$.

5. Indicate whether each of the following relations on the given set is reflexive, symmetric, antisymmetric, or transitive. Explain your answers.

(a) $A = \{x \in \mathbb{R} : x > 0\}$ and $x\rho y$ iff $xy = 0$.
(b) $B = \{x \in \mathbb{Z} : x \geq 0\}$ and $x\rho y$ iff $x + y = 4$.
(c) $C = \{1, 2, 4\}$ and $\rho = \{(2,1), (2,4), (1,1), (4,2), (2,2)\}$.
(d) $D = \mathbb{Q}$ and $x\rho y$ iff $x - y \in \mathbb{Z}$.

6. Define a relation $\preceq$ on $\mathbb{N}$ by declaring that for $x, y \in \mathbb{N}$,

$$x \preceq y \iff (x = y) \vee (2x \leq y).$$

Show that $\preceq$ is a partial order on $\mathbb{N}$, but not a total order.

7. Let $A = \{1, 2, 3 \ldots, 18\}$ and $B = \{\, n \in \mathbb{N} : n \mid 18 \,\}$.

(a) Draw lattice diagrams for $(A, |)$ and $(B, |)$.
(b) Indicate any maximal, minimal, greatest and least elements in each.
(c) Find the least upper bound for $\{1, 2, 3\}$ in B.
(d) Find a subset of A which has no least upper bound.

8. Let $(A, \preceq)$ be a poset with the least upper bound property. Let $S \subseteq A$ with $S \neq \emptyset$. Suppose S has at least one lower bound. Put $L_S = \{ b \in A : b \text{ is a lower bound for } S \}$. Show that L_S is nonempty and has at least one upper bound.

From the least upper bound property, we know that $\sup L_S$ exists. Put $g = \sup L_S$. Show that g is a *greatest lower bound* for S, in other words that

- g is a lower bound for S, and
- if b is a lower bound for S then $b \preceq g$.

9. Define a relation ρ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by declaring that, for $(x, y), (u, v) \in \mathbb{R}^2$,

$$(x, y) \ \rho \ (u, v) \iff |\, x \,| + |\, y \,| = |\, u \,| + |\, v \,| \ .$$

Show that ρ is an equivalence relation. What is the equivalence class $T_{(x,y)}$ of the element (x, y)?