NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. Which of the following sentences are statements, which are predicates, and which are neither? Translate all the statements and predicates into symbols. If the statement is true prove it. If it is false give a counterexample.

(a) 19 is a prime number. (I.e. It has no factor other than itself and 1).

(b) If n is even then n is not prime.

(c) Is 17 a prime number?

(d) Solve $x^2 - 4 = 0$.

(e) Every even number is the sum of two odd numbers.

(f) There exists a real number n such that for all real r, $r.n = n.r = n$.

Express the last result in terms of quantifiers.

2. Let A and B be statements. Construct truth tables for the following statements. For each statement, state whether it is a tautology, a contradiction or neither.

(a) $(\sim A \vee B) \wedge \sim(A \implies B)$.

(b) $\sim(A \wedge B) \implies \sim(A \vee B)$.

(c) $(A \implies B) \iff (A \vee \sim B)$.

(d) $(\sim A \implies B) \implies (\sim B \implies A)$.

3. For any integer n, let $A(n)$ be the statement

"If n is a multiple of 4 then $n^2 + 1$ is odd".

(a) Write down the contrapositive of $A(n)$.

(b) Write down the converse of $A(n)$.

(c) Write down the negation of $A(n)$.

(d) Is $A(n)$ true for some $n \in \mathbb{N}$? If so, give an example, if not give a proof.

(e) Is $A(n)$ true for every $n \in \mathbb{N}$? If so, give a proof, if not give a counterexample.

(f) Is the contrapositive of $A(n)$ true for some $n \in \mathbb{N}$? Is it true for all $n \in \mathbb{N}$? Give brief reasons for your answer.

(g) Is the converse of $A(n)$ true for some $n \in \mathbb{N}$? If so, give an example, if not give a proof.

(h) Is the converse of $A(n)$ true for all $n \in \mathbb{N}$? If so, give a proof, if not give a counterexample.

4. Use a proof by contradiction to show that the cube root of 3 is not a rational number.

5. Define the function $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) = n^3 + n$.

(a) Use a direct proof to show that if $m, n \in \mathbb{N}$ with $m < n$ then $f(m) < f(n)$.

(b) Use proof by contraposition to show that if $m, n \in \mathbb{N}$ with $f(m) < f(n)$ then $m < n$.

(c) A function $g : A \to B$ is *one-to-one* if, for every $a, b \in A$, if $g(a) = g(b)$ then $a = b$. Use proof by contradiction to show that f is one-to-one.

6. (**23 marks**) Let A, B and C be sets.

(a) Give a counterexample to the statement $(A \cup B) \setminus (A \cap B) = A \setminus B$.

(b) If $B \subseteq C$ then show that $(B \setminus A) \subseteq (C \setminus A)$.

(c) Show that $A \subseteq B \iff A \setminus B = \Phi$.

(d) Show that $A \subseteq B \iff (A \cup B) = (B \setminus A) \cup (A \cap B)$.

(e) Let Λ be a non-empty indexing set and B_α a set for each $\alpha \in \Lambda$. Show that

$$A \setminus (\bigcup_{\alpha \in \Lambda} B_\alpha) = \bigcap_{\alpha \in \Lambda} (A \setminus B_\alpha).$$

7. (**12 marks**) Let $A = \{1, 2, 4, 5\}$ and let $B = \{3, 5, 6\}$. Find

(a) The number of elements in $\mathcal{P}(A)$.

(b) $\mathcal{P}(B)$.

(c) $\mathcal{P}(A \cap B)$.

(d) The number of elements in $\mathcal{P}(A \cup B)$.