Monday: Orderings

Properties of relations [4.1]

Example 1 (4.1.10). *Consider the following relations. Which, if any, of the four properties reflexive, symmetric, antisymmetric and transitive, do these relations have?*

1. $A = \mathbb{N}$, $x \sim y$ *if* $x + y$ *is even.*
2. $A = \mathbb{N}$, $x \sim y$ *if* $x + y$ *is odd.*
3. $A = \mathcal{P}(\mathbb{N})$, $x \sim y$ *if* $x \subseteq y$.
4. $A = \mathbb{R}$, $x \sim y$ *if* $x = 2y$.
5. $A = \mathbb{R}$, $x \sim y$ *if* $x - y$ *is rational.*

Example 2. *Recall that for* $m, n \in \mathbb{Z}$, $m \mid n$ *if there is some* $a \in \mathbb{Z}$ *with* $n = ma$. *Show that* $\mid$ *is reflexive and transitive and not symmetric. Show that* $\mid$ *is not antisymmetric on* $\mathbb{Z}$ *but is antisymmetric on* $\mathbb{N}$.

Partial and total orderings [4.2]

Definition. *A relation* ρ *on a set* A *is a* partial order *(or* partial ordering*) if* ρ *is reflexive, antisymmetric and transitive.*

If ρ *is a partial order on* A, *we say that* A *is a* partially ordered set under ρ, *or that the pair* (A, ρ) *is a paritally ordered set. We often abbreviate "partially ordered set" to* poset.

Example 3. *From Example 2 we see that* $\mid$ *is a partial order on* $\mathbb{N}$.

Example 4. *Let* A *be a set, and let* $S = \mathcal{P}(A)$. *Then* S *is a poset under the relation* $\subseteq$.

We have the convention of using the symbol $\leq$ for a typical partial order. Sometimes we will use the symbol $\preceq$ instead, particularly when the usual order on the real numbers is being used in the definition of our order.

Example 5. *Let* $A = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, *and define a relation* $\preceq$ *on* A *by declaring that*

$$(u, v) \preceq (x, y) \iff u \leq x \land v \leq y.$$

Then $\preceq$ *is a partial order.*

Proof. We must show $\preceq$ is reflexive, antisymmetric and transitive.

Reflexive: Let $(x, y) \in A$. Then $x \leq x$ and $y \leq y$ so $(x, y) \preceq (x, y)$.

Antisymmetric: Let $(x, y), (u, v) \in A$ with $(x, y) \preceq (u, v)$ and $(u, v) \preceq (x, y)$. Then $x \leq u$ and $y \leq v$, and $u \leq x$ and $v \leq y$. Combining these we have $x \leq u \leq x$, so $x = u$, and $y \leq v \leq y$, so $y = v$. Hence $(x, y) = (u, v)$, as required.

Transitive: Let $(x, y), (u, v), (z, w) \in A$ with $(x, y) \preceq (u, v)$ and $(u, v) \preceq (z, w)$. Then $x \leq u$ and $y \leq v$, and $u \leq z$ and $v \leq w$. Combining these we have $x \leq u \leq z$, so $x \leq z$, and $y \leq v \leq w$, so $y \leq w$. Hence $(x, y) \preceq (z, w)$, as required.

□

The name "partial" order is supposed to allow the possibility that not all the elements are ranked in the order: for example, in the previous example we have pairs $(1, 5)$ and $(3, 3)$ which are *incomparable*: we have both $(1, 5) \not\preceq (3, 3)$ and $(3, 3) \not\preceq (1, 5)$ (the first is because $5 \not\leq 3$ and the second because $3 \not\leq 1$).

Definition. *A* total order *or* total ordering *on a set A is a partial order $\preceq$ on A with the additional property that for any $a, b \in A$ we have $a \preceq b$ or $b \preceq a$.*

Tuesday: Lattice diagrams, maximal elements and greatest elements

Lattice diagrams [4.2]

A good way to represent a (reasonably small) poset is with a *lattice diagram*. To understand these, first we need some notation. If $\preceq$ is a partial order on A, we define a new relation $\prec$ on A by

$$x \prec y \iff x \preceq y \wedge x \neq y.$$

Similarly, if $\leq$ is a partial order on A we define a new relation $<$ by declaring that $x < y \iff x \leq y$ and $x \neq y$.

Definition. *Let $\preceq$ be a partial order on A, and let $a, b \in A$. Then b* covers *a if $a \prec b$ and there does not exist any $c \in A$ with $a \prec c \prec b$.*

Example 6. *In the poset $\mathbb{Z}$, with its usual ordering, $n + 1$ covers n for each n.*

To draw a lattice diagram for a poset A, we put a blob for each element of A, and an edge going up from a to b if b covers a. We can then see that for $a, b \in A$, $a \prec b$ if there is a path going upwards from a to b.

Example 7. *Let $A = \{ n \in \mathbb{N} : n \mid 30 \} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. Draw a lattice diagram for the poset $(A, |)$.*

Example 8. *Draw a lattice diagram for the poset $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$.*

Maximal and minimal, greatest and least elements [4.2]

Definition. *Let $(A, \leq)$ be a poset, and let $a \in A$. Then a is* maximal *if there is no $b \in A$ with $a < b$, and* minimal *if there is no $b \in A$ with $b < a$. Further, a is a* greatest *element if $b \leq a$ for all $b \in A$ and a* least *element if $a \leq b$ for all $b \in A$.*

Example 9. *Draw a lattice diagram for (A, mid), where $A = \{1, 2, 3, \ldots, 10\}$. Identify any maximal and minimal elements and any greatest and least elements.*

Theorem 10. *If a poset A has a greatest element, then it is unique. Likewise, if A has a least element then it is unique.*

Note that we can find examples where A has no greatest or least element, and we can find examples where there are more one maximal elements or minimal elements.

Every **finite** partial order has at least one maximal element and one minimal element. However, the poset $(\mathbb{Z}, \leq)$ has no maximal or minimal elements.

Thursday: Least upper bounds and partitions

Upper bounds and lower bounds [4.2]

Definition. *Let A be a poset and $S \subseteq A$. Then u is an* upper bound for S *if* $(\forall a \in S)(a \leq u)$*, and l is a* lower bound *for S if* $(\forall a \in S)(l \leq a)$.

Example 11. *Let $A = \{1, 2, \ldots, 10\}$, partially ordered by $|$. Find any upper and lower bounds of the sets* $\{2, 3\}$, $\{2, 5\}$ *and* $\{3, 5\}$.

Definition. *Let A be a poset and let $S \subseteq A$. An element $s \in A$ is a* least upper bound *or* supremum *for S if*

- *s is an upper bound for S, and*
- *if u is also an upper bound for S then $s \leq u$.*

If S has a supremum, then the supremum is unique. However, there are some partial orders with subsets which have no supremum, either because S has no upper bounds at all or because there is no least element in the set of upper bounds of S.

Definition. *Let A be a poset. We say that A has the* least upper bound property *if every nonempty subset of A with at least one upper bound has a least upper bound.*

Example 12. *The poset $(\mathbb{R}, \leq)$ has the least upper bound property: this is one of the axioms for the real numbers we will see in Chapter 8.*

Example 13. *The poset $(\mathbb{N}, |)$ has the least upper bound property: we will prove this in Chapter 6.*

Partitions [4.3]

Definition. *Let S be a set. Then Ω is a* partition *of S if*

- $\Omega \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$*; and*
- *if $A, B \in \Omega$ with $A \neq B$ then $A \cap B = \emptyset$; and*
- $S = \bigcup_{A \in \Omega} A$.

In other words, Ω is a pairwise disjoin family of nonempty subsets of S such that every element of S is in one of the sets in Ω.

Example 14. *Let* $O = \{ 2n+1 : n \in \mathbb{Z} \}$, $E = \{ 2n : n \in \mathbb{Z} \}$. *Then* $\Omega = \{O, E\}$ *is a partition of* $\mathbb{Z}$.

Example 15. *The family* $\{ [n, n+1) : n \in \mathbb{Z} \}$ *is a partition of* $\mathbb{R}$.

Example 16. *Find all possible partitions of* $\{1, 2, 3\}$.

Friday: Equivalence relations

The set of relatives

Let $\sim$ be a relation on a set A. For $a \in A$, define the set of relatives of a to be

$$T_a = \{ x \in A : a \sim x \}.$$

Notice that $\sim$ is reflexive iff $a \in T_a$ for all $a \in A$. So if *sim* is reflexive then $\Omega = \{ T_a : a \in A \}$ will be a family of nonempty subsets of A whose union is A.

Definition. *An* equivalence relation *on a set A is a relation on A which is reflexive, symmetric and transitive.*

Example 17. *Define* $\sim$ *on* $\mathbb{Z}$ *by declaring that for* $m, n \in \mathbb{Z}$ *we have*

$$m \sim n \iff 5 \mid m - n.$$

Then $\sim$ *is an equivalence relation.*

Theorem 18. *If* $\sim$ *is an equivalence relation on* A *and* $a, b \in A$ *then the following are equivalent:*

1. $a \sim b$.
2. $T_a \cap T_b \neq \emptyset$.
3. $T_a = t_b$.

Theorem 19. *If* $\sim$ *is an equivalence relation on* A *then* $\Omega = \{ T_a : a \in A \}$ *is a partition of* A.

Conversely, if Ω *is a partition of* A *then the relation* $\sim$ *defined by declaring that*

$$x \sim y \iff (\exists A \in \Omega)(a \in A \wedge b \in A)$$

is an equivalence relation and $\Omega = \{ T_a : a \in A \}$.