Monday: Differentiability

In today's lecture we will learn exactly what it means for a function to be differentiable. Before doing that, we will find a little more about limits.

Limits of products and quotients

Theorem 1. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ be functions, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist then $\lim_{x\to c} f(x)g(x)$ exists and is equal to $\lim_{x\to c} f(x) \lim_{x\to c} g(x)$.*

Proof. Let $F = \lim_{x\to c} f(x)$ and $G = \lim_{x\to c} g(x)$. Put $\eta = \frac{\varepsilon}{|G|+\frac{1}{2}+|F|}$. Choose $\delta_1\delta_2 > 0$ so that if $0 < |x-c| < \delta_1$ then $|f(x) - F| < \eta$ and if $x \in A$ with $0 < |x-c| < \delta_2$ then $|g(x) - G| < \min\{\eta, \frac{1}{2}\}$. Note that if $x \in A$ with $0 < |x-c| < \delta_2$ then $|g(x) - G| < \frac{1}{2}$ so $|g(x) < |G| + \frac{1}{2}$. Put $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in A$ with $1 < |x-c| < \delta$. Then

$$
\begin{aligned}
|f(x)g(x) - FG| &= |f(x)g(x) - Fg(x) + Fg(x) - FG| \\
&\le |f(x)g(x) - Fg(x)| + |Fg(x) - FG| && \text{(triangle inequality)} \\
&= |f(x) - F||g(x)| + |F||g(x) - G| \\
&\le \eta(|G| + \frac{1}{2}) + |F|\eta \\
&= \varepsilon
\end{aligned}
$$

Thus $\lim_{x\to c} f(x)g(x)$ exists and equals FG. $\square$

Note that we might be a little lazy and write this as "$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$". However, we must remember that limits need not exist, and the existence of the limit is part of the assertion. Also the converse does not hold: it is quite possible for $\lim_{x\to c} f(x)g(x)$ but neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ to exist.

Theorem 2. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ exists and is non-zero then $\lim_{x\to c} \frac{1}{f(x)}$ exists and is equal to $\frac{1}{\lim_{x\to c} f(x)}$.*

Proof. Exercise. Note that we need to choose δ small enough to ensure that $f(x)$ is non-zero within a distance of δ from c: in fact we will want to ensure that $\frac{1}{f(x)}$ does not get too large—say, does not get larger than $2F$ where $F = \lim_{x\to c} f(x)$—so we will choose δ small enough to ensure that $|f(x) - F| < \frac{F}{2}$, which ensures that $|f(x)| > |F - \frac{F}{2}|$. See the proof that if $b_n \to B \neq 0$ then $\frac{1}{b_n} \to \frac{1}{B}$ in the notes for week 11 for more ideas. $\square$

Exercise 3. *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$ be functions, and let c be an accumulation point of A. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist then $\lim_{x\to c} f(x) + g(x)$ exists and is equal to $\lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.*

Differentiability

Definition. *Let $A \subseteq \mathbb{R}$. If $c \in A$, we say that c is an* interior point *of A if there is some $\varepsilon > 0$ such that $B_\varepsilon(c) \subseteq A$. We denote the set of interior points of A by $\mathrm{int}(A)$.*

Thus A is open if and only if every point of A is an interior point of A, i.e. if $A = \mathrm{int}(A)$.

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $c \in \mathrm{int}(A)$. We say that f is* differentiable at c *if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, or equivalently if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. If the limit exists, we denote it by $f'(c)$, and call this number the* derivative of f at c. *For $S \subseteq \mathrm{int}(A)$ we say that f is* differentiable on S *if f is differentiable at all $c \in S$. When A is open we say that f is* differentiable *if it is differentiable on A.*

Example 4. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is differentiable and, for all $c \in \mathbb{R}$, $f'(c) = 2c$.*

Proof. For all $h \neq 0$ we have

$$\begin{aligned}
\frac{f(c+h) - f(c)}{h} &= \frac{(c+h)^2 - c^2}{h} \\
&= \frac{c^2 + 2ch + h^2 - c^2}{h} \\
&= \frac{2ch + h}{h} \\
&= 2c + h
\end{aligned}$$

Now $\lim_{h\to 0} 2c + h = 2c$, so $f'(c)$ exists and equals $2c$, as required. □

Theorem 5. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $c \in \mathrm{int}(A)$. If f is differentiable at c then f is continuous at c.*

Proof. Suppose f is differentiable at c. Then $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists and equals $f'(c)$. We also have $\lim_{x\to c}(x - c)$ exists and equals 0. So by Theorem 1, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x - c)$ exists and equals $f'(c) \cdot 0 = 0$. So $\lim_{x\to c}(f(x) - f(c) = 0$, so $\lim_{x\to c} f(x) = f(c)$, so f is continuous at c. □

Tuesday: Rolle's Theorem and the Mean Value Theorem

Lemma 6. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains its bounds. In other words, there exist $c, d \in [a, b]$ with $f(c) = \max\{ f(x) : x \in [a, b] \}$ and $f(d) = \min\{ f(x) : x \in [a, b] \}$.*

Proof. First we will show that f is bounded. Suppose, for a contradiction, that f is unbounded above. For each $n \in \mathbb{N}$ we can find some $x_n \in [a, b]$ with $f(x_n) > n$. Now the sequence (x_n) is bounded, so it has a convergent subsequence, (x_{i_n}) say. Let x be the limit of this subsequence. Since (x_n) is a sequence in $[a, b]$, which is closed, we have $x \in [a, b]$. But then $f(x_{i_n})$ converges to $f(x)$, which is impossible because $(f(x_{i_n}))$ is an unbounded sequence.

Similarly, f is bounded below.

Put $s = \sup\{ f(x) : x \in [a, b] \}$. For every n, there is some $y_n \in [a, b]$ with $s - \frac{1}{n} < f(y_n) \leq s$. Then (y_n) is a bounded sequence in $[a, b]$, so it has a subsequence (y_{j_n}) which converges to some $c \in [a, b]$. By continuity, $(f(y_{j,n}))$ converges to $f(c)$. But, by construction, $(f(y_{j_n})$ converges to s. So $f(c) = s = \sup\{ f(x) : x \in [a, b] \}$. So $f(c) = \max\{ f(x) : x \in [a, b] \}$.

Similarly, f attains its infimum. $\square$

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Then a is a* local maximum *of f if there is some $\varepsilon > 0$ such that for all $x \in A$ with $|x - a| < \varepsilon$ we have $f(x) \leq f(a)$. Similarly, a is a* local minimum *of f if there is some $\varepsilon > 0$ such that for all $x \in A$ with $|x - a| < \varepsilon$ we have $f(x) \geq f(a)$.*

Theorem 7. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function and let $a \in \operatorname{int} A$. If $f'(a)$ exists and a is a local maximum or local minimum of f then $f'(a) = 0$.*

Proof. Exercise. $\square$

Note that we need both $f'(a)$ exists and $a \in \operatorname{int}(A)$ as hypotheses here: consider the examples $f : [0, 1] \to \mathbb{R}$ given by $f(x) = x$, which has 1 as a local maximum, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = |x|$ which has 0 as a local minimum.

Theorem 8 (Rolle's Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there is some $c \in (a, b)$ with $f'(c) = 0$.*

Proof. Put $k = f(a) = f(b)$. We know that f is continuous on $[a, b]$ so it attains its maximum at some $c \in [a, b]$. Suppose first that $c \neq a$ and $c \neq b$. Then $c \in (a, b)$, so c is a local maximum of f, and $f'(c)$ exists, so by the previous result we must have $f'(c) = 0$.

Similarly, we know that f attains its minimum at some $d \in [a, b]$, and if $d \neq a, b$ then $d \in (a, b)$ and $f'(d) = 0$.

The only remaining possibility is that $c, d \in \{a, b\}$. But then we must have $f(x) = k$ for all $x \in [a, b]$, so $f'(x) = 0$ for all $x \in (a, b)$. $\square$

Theorem 9 (Mean Value Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Then there is some $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Put $k = \frac{f(b)-f(a)}{b-a}$ and define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - kx$. Then g is continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) = f'(x) - k$. Also

$$\begin{aligned} g(b) - g(a) &= f(b) - kb - f(a) - ka \\ &= (f(b) - f(a)) - k(b - a) \\ &= (f(b) - f(a)) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= 0, \end{aligned}$$

so $g(a) = g(b)$. Hence by Rolle's Theorem there is some $c \in (a, b)$ with $g'(c) = 0$. But then $f'(c) - k = 0$, so $f'(c) = k$, as required. $\square$